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CX. *The Initial Motion of a Projectile.* By E. T. Hanson, *B.A.**

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Introduction.

THE problem of a projectile's motion first attracted my attention some years ago in connexion with the theory of prediction in anti-aircraft gunnery. I did not, however, proceed at that time beyond obtaining the equations of motion under an arbitrary system of impressed forces.

Recently, upon reading a very interesting paper entitled "The Aerodynamics of a Spinning Shell," Phil. Trans. Series A, vol. ccxxi., it occurred to me that the following application might be made of the force system, therein obtained by an ingenious combination of theory and experiment. The application referred to is based upon the remarkably simple forms which the most important forces acting upon a particular shell appear to take at high velocities. These forms are found to be applicable to the motion of the shell during the first seven or eight seconds of its time of flight, that is to say, during its "settling-down" period.

The shell mentioned is the 16 lb. 3-inch A.A. shell, but it may reasonably be assumed that shells of similar form will

* Communicated by R. H. Fowler, M.A., of Trinity College, Cambridge.

be accompanied by corresponding simplicity in the forces which act upon them.

In Section 1, I have outlined the proof of the general equations of motion as I originally obtained them, and also the modified equations which are applicable to the problem under discussion. These equations are obtained by a different method in the paper referred to; hardly more is given, therefore, than is necessary to explain the notation used.

In Section 2, I have discussed shortly the initial portion of the plane trajectory, and the system of forces acting upon the shell over this portion.

In Section 3, I have considered the solution of the equations of motion during the period over which the force system is valid.

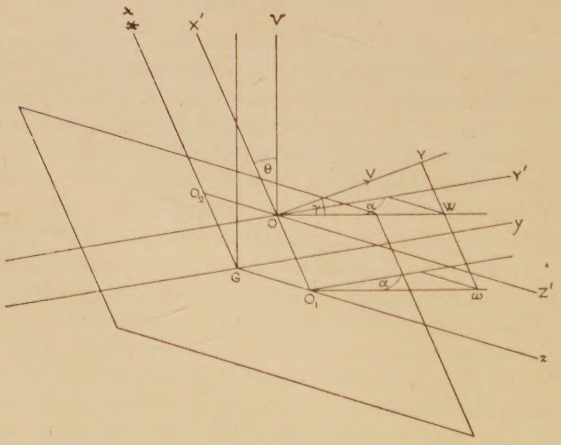
Section 1.—The General Equations of Motion.

The Kinematics of the Motion.

(a) *The motion of the centre of gravity.*

This motion is obtained relatively to a predetermined curve in space, which is known as the plane trajectory.

Fig. 1.



Let O (fig. 1) be the position of the centre of gravity of the projectile at any instant. Through O construct a plane perpendicular to the plane trajectory, and cutting the latter in G.

Let Gy be the tangent to the plane trajectory at G, Gx the normal, and Gz perpendicular to the plane of the plane trajectory. Through O draw axes parallel to Gx, Gy, Gz.

Let θ be the angle between the vertical VO and Gx or OX'.

The motion of the centre of gravity O is referred to co-ordinate axes Gx and Gz in a moving plane which at every instant is perpendicular to the plane trajectory. This plane always contains O and must rotate about OZ' with angular velocity $\dot{\theta}$.

The coordinates x and z of O are convenient for determining the drift, but for the discussion of the actual motion of the centre of gravity other coordinates are more useful.

Let OY be the resultant direction of motion of the centre of gravity at the instant in question, and let the plane X'OY cut the plane Y'OZ' in OW.

$$\text{Let} \quad Y'OW = \alpha$$
$$\text{and} \quad YOW = \gamma.$$

Let v be the resultant velocity of the centre of gravity along OY. The velocity of G along the tangent to the plane trajectory is

$$u = v\cos\alpha\cos\gamma - OO_1 . \dot{\theta}. \quad . \quad . \quad . \quad . \quad (1)$$

Imagine a plane X'OO_1wW to move with O in such a manner that O_1w is at O_1 the tangent to a curve to be determined by the value of $\dot{\alpha}$ and $\frac{\partial}{\partial t}(v\cos\gamma)$.

This curve would be plotted on a plane which can be considered to roll on the plane trajectory round the moving line Gz. In this way a cylindrical surface is constructed which always contains the centre of gravity.

In this cylindrical surface is plotted a curve to be determined by the values of v and $\dot{\gamma}$. The common origin of both these curves is coincident with that of the plane trajectory. The values found for v, $\dot{\alpha}$, and $\dot{\gamma}$ are sufficient to determine the whole motion of the centre of gravity.

(b) *The motion about the centre of gravity.*

In fig. 2 OXYZ is a system of rectangular axes drawn through the centre of gravity O of the projectile.

In the consideration of this part of the motion it is postulated :—

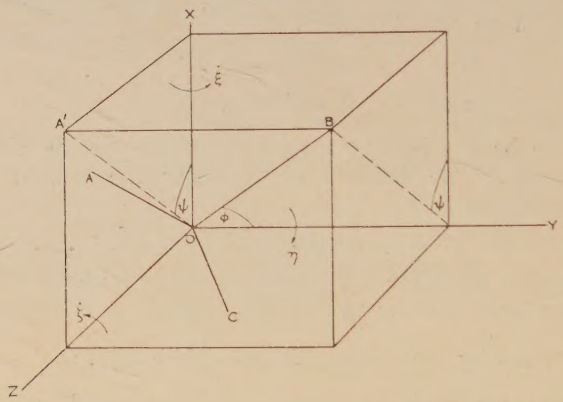
(1) That at the instant under consideration the centre of gravity O is moving in the direction OY.

(2) That the axes OX, OY, OZ are instantaneously turning about themselves with angular velocities ξ, η, ζ, which are right-handed on looking along OX, OY, and OZ respectively from the origin.

Let OB be the axis of the projectile, making an angle ϕ with OY. ϕ is called the "yaw" of the shell and BOY is the plane of yaw, which makes an angle ψ with the plane XOY. ψ is the precession of the shell with respect to the plane XOY.

OABC is a system of rectangular axes, of which OA lies in the plane of yaw. Let C be the moment of inertia of the shell about its axis of spin OB. Let A be that about any transverse axis, since the shell is symmetrical about its axis

Fig. 2.



of spin. Let ω be the spin of the projectile relatively to the moving plane AOB.

Then the motion of the projectile about its centre of gravity relatively to the system OXYZ is completely determined by ϕ, ψ, and ω, these being precisely the same quantities as are commonly employed in the theory of the ordinary spinning-top, where the system OXYZ is fixed.

Let the total instantaneous angular velocities of the projectile about OB, OA, OC be respectively Ω, Ω_1, Ω_2.

The system of axes OXYZ is determined in every position by the following convention :—

As the centre of gravity moves the plane XOY always contains OX′.

Let the direction cosines of OY with respect to OX′, Y′, Z′ be l_2, m_2, n_2.

The general values of $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ are

$$\left.\begin{aligned} \dot{\xi} &= -\dot{\alpha}\cos\gamma - \dot{\theta}\sin\gamma\sin\alpha, \\ \dot{\eta} &= -\dot{\alpha}\sin\gamma + \dot{\theta}\cos\gamma\sin\alpha, \\ \dot{\zeta} &= \quad \dot{\gamma} \qquad + \dot{\theta}\cos\alpha. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

The Dynamical Equations.

The motion of the centre of gravity.

The forces acting upon the centre of gravity are :—

(1) The drag R acting along YO and directly opposed to the direction of motion.

(2) A force F_1 acting in the plane of yaw, perpendicularly to OY, and in the direction OA′ (fig. 2).

Let m be the mass of the projectile, and v its velocity.
Let angle YOV = β.
Since the centre of gravity is moving along OY with velocity v,

$$m\frac{\partial v}{\partial t} = -\mathrm{R} - mg\cos\beta$$
$$= -\mathrm{R} - mg[\cos\theta\sin\gamma + \sin\theta\cos\gamma\cos\alpha]. \quad (3)$$

Since the motion in the directions OX′ and OZ′ is referred to axes rotating about OZ′ with angular velocity $\dot{\theta}$, the equations of motion parallel to OX′ and OZ′ are respectively

$$m\left\{\frac{\partial}{\partial t}(l_2 v) + m_2 v\dot{\theta}\right\} = -\mathrm{R}l_2 - mg\cos\theta + \mathrm{F}_1\cos\psi\cos\gamma, \quad (4)$$

and

$$m\frac{\partial}{\partial t}(n_2 v) = -\mathrm{R}n_2 + \mathrm{F}_1\sin\psi\cos\alpha - \mathrm{F}_1\cos\psi\sin\gamma\sin\alpha, \quad (5)$$

l_2, m_2, n_2 being subsequently expressed in terms of α and γ.

The motion about the centre of gravity.

The total kinetic energy of rotation is

$$\mathrm{T} = \tfrac{1}{2}[\mathrm{C}\Omega^2 + \mathrm{A}(\Omega_1^2 + \Omega_2^2)], \quad (6)$$

in which, as can be readily verified,

$$\left.\begin{aligned}\Omega &= \omega + (\dot{\psi} + \dot{\eta})\cos\phi + (\dot{\zeta}\sin\psi + \dot{\xi}\cos\psi)\sin\phi,\\ \Omega_1 &= (\dot{\psi} + \dot{\eta})\sin\phi - (\dot{\zeta}\sin\psi + \dot{\xi}\cos\psi)\cos\phi,\\ \Omega_2 &= \dot{\phi} + \dot{\zeta}\cos\psi - \dot{\xi}\sin\psi.\end{aligned}\right\} \quad (7)$$

It is assumed that there are three couples M_0, M_1, and M_2 acting about OB, OA, and OC respectively.
Lagrange's equations are extremely complicated, when

expanded, but the following equivalent equations are concise and convenient:—

$$\left.\begin{aligned} C\frac{\partial\Omega}{\partial t} &= M_0, \\ \frac{1}{\sin\phi}\left[A\sin\phi\frac{\partial\Omega_1}{\partial t}-(C\Omega\sin\phi-A\Omega_1\cos\phi)\Omega_2+A\Omega_2\frac{\partial\Omega_2}{\partial\psi}\right] &= M_1 \\ A\frac{\partial\Omega_2}{\partial t}+C\Omega\Omega_1-A\Omega_1\frac{\partial\Omega_1}{\partial\phi} &= M_2, \end{aligned}\right\} \quad (8)$$

in which the differential coefficients with respect to ϕ and ψ are partial.

The Initial Motion.

In considering the initial motion of a shell ϕ may be taken as small. $\dot{\xi}$, $\dot{\zeta}$, and $\dot{\theta}$ are also small, so that the squares and products of all these quantities may be neglected.

In Section 2 the values of M_0, M_1, M_2, F_1 are given. The following abbreviations are convenient:—

$$\left.\begin{array}{lll} \dfrac{C\Omega}{A}=P & \dfrac{F'}{mv}=F & \dfrac{N'}{A}=N \\ \phi e^{i\psi}=u & -\ddot{\theta}+iP\dot{\theta}-L\dot{\theta}=iPH & \dfrac{L'}{A}=L \\ Q=\zeta-i\xi & & \dfrac{M'}{A}=M_1. \end{array}\right\} \quad (9)$$

The last two equations of (8) then combine into the complex equation

$$\frac{\partial^2}{\partial t^2}(u+Q)-(iP-L)\frac{\partial}{\partial t}(u+Q)-(M_1+iN)u=iPH, \quad (10)$$

and the equations of motion of the centre of gravity combine into

$$\frac{\partial}{\partial t}\left(\frac{Q}{\cos\theta}\right)=\frac{Fu}{\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Equations (10) and (11) determine the whole motion, when ϕ is small.

The motion of the centre of gravity is given by the formula

$$Q=\frac{\dot{x}+i\dot{z}}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Section 2.—The Force System.

As mentioned in the introduction, the forces acting upon the shell take remarkably simple forms when the velocity exceeds 1000 f.s. In this section the force system will be formulated in accordance with this observation.

An interesting paper by Mallock upon "Air Resistance" appeared in the Royal Soc. Proc. vol. lxxix., and was followed in the same volume by a paper upon "The Ranges and Behaviour of Rifled Projectiles in Air."

In the latter paper he applies a very simple law of resistance in the discussion of the motion of a shell with a very flat trajectory.

This law of resistance applies to projectiles moving with speeds varying from about the velocity of sound to above three times that velocity.

The most important force acting through the centre of gravity is the "drag," and it is remarkable that the resistance curve for pointed projectiles, where resistance is plotted against velocities ranging from about 1000 f.s. to 3000 f.s., is almost a straight line cutting the abscissa at the point representing 850 f.s. and the 3000 f.s. ordinate at the point representing 40 lb. per square inch of the cross section of the shell.

Let V_0 = the muzzle velocity of the shell.
v = the velocity of the shell at any instant.
u_0 = 850 f.s.
W = weight of shell in lb.
m = mass of shell.
R = drag.
f = retardation of shell.
d = diameter of shell in inches.
R′ = resistance in lb. per square foot of cross-sectional area of the shell.

It is found that the resistance is given by the equation

$$\left.\begin{aligned} &\mathrm{R}' = 2{\cdot}53\,(\mathrm{V} - u_0). \\ \text{Let}\quad &\lambda = \frac{2{\cdot}53\,\pi d^2 g}{4 \times 144\,.\,\mathrm{W}}. \\ \text{Then}\quad &\mathrm{R} = mf = \frac{\mathrm{W}}{g} f, \\ \text{and}\quad &f = \lambda(v - u_0). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

This simple equation for the retardation represents the motion very accurately.

When the velocity of the shell exceeds that of sound, the forces and couples may be expressed very approximately as follows :—

(1) A couple M_2 tending to increase ϕ.

$$M_2 = \left[\frac{\kappa}{m\lambda} R + \kappa(u_0 - u_1)\right] \sin \phi,$$

where u_1 is a constant depending upon the nature of the shell, and $u_0 - u_1$ is small.

This couple is important in the production of drift.

Write

$$M_2 = M' \sin \phi.$$

(2) A force F_1 acting along OA′ through the centre of gravity and in the plane of yaw.

$$F_1 = \frac{\kappa' M v}{v_0 - u_0} \cdot \sin \phi,$$

where κ' is a constant depending upon the form of the shell.

Write

$$F_1 = F' \sin \phi.$$

(3) A couple M_1 acting about OA, tending to increase the rate of precession.

This couple is small compared with M_2 but it has an important effect upon the stability of the shell.

Write

$$M_1 = N' \sin \phi.$$

(4) A couple tending to retard both the increase of ϕ and the rate of precession. This couple has two components proportional to Ω_1 and Ω_2 respectively. These components will be written $L'\Omega_1$ and $L'\Omega_2$, the coefficient L' being the same in both owing to the symmetry of the shell about its axis.

(5) The air-friction couple tending to reduce the spin can be neglected over the initial portion of the trajectory.

Until the velocity of the shell drops to about that of sound the above relations connecting the forces, which act upon the shell, have been shown by experiment on a particular form of shell to be the true relations with a considerable degree of certainty.

Experiment appears to show that the shell settles down to a motion in accordance with theory in a small fraction of a second after it leaves the muzzle of the gun. It is, therefore, highly probable that the following theory gives a very accurate picture of the complete motion of the shell at all

angles of fire during a period of its time of flight equal to seven or eight seconds. Beyond this time in high angle fire the curvature of the trajectory begins to become effective in modifying the motion of the shell. Furthermore, the force system ceases to obey the above laws. A step by step method for the integration of the equations of motion has then to be adopted.

In the case, however, of high velocity shells and low trajectories, which are common in naval gun-fire, the theory presents a very complete solution. For in these cases the force system is valid throughout the whole time of flight.

In order to establish the simple law of resistance in high angle fire a short discussion is given of the initial portion of the plane trajectory for the 3 inch 20 cwt. anti-aircraft gun, firing a 16 lb. shell. In this case

$$V_0 = 2000 \text{ f.s.}$$
$$\lambda = \tfrac{1}{4}.$$

The Plane Trajectory.

The Equations of the trajectory under the law of retardation, $f = \lambda(v - u_0)$.

It is assumed that the axis of the projectile is always very nearly in the direction of the tangent to the trajectory.

The resistance of the air is therefore effectively tangent to the trajectory.

Then for the equations of motion we have

$$\left.\begin{aligned} \ddot{x} &= -\lambda(v-u_0)\cos\theta, \\ \ddot{y} &= -g-\lambda(v-u_0)\sin\theta, \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

where θ is the angle which the tangent to the trajectory makes with the horizontal.

Now in the applications of the theory the departure of θ from its initial value θ_0 is never very great. It is, therefore, tentatively assumed that throughout the flight

$$\left.\begin{aligned} u_0\cos\theta &= u_0\cos\theta_0, \\ \text{and} \qquad u_0\sin\theta &= u_0\sin\theta_0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15)$$

Integration of (14) gives

$$\left.\begin{aligned} x &= u_0\cos\theta_0 \,.\, t + \frac{1}{\lambda}(v_0-u_0)\cos\theta_0(1-e^{-\lambda t}), \\ y &= \left(u_0\sin\theta_0 - \frac{g}{\lambda}\right)t + \frac{1}{\lambda}\left[(v_0-u_0)\sin\theta_0 + \frac{g}{\lambda}\right](1-e^{-\lambda t}). \end{aligned}\right\} \quad (16)$$

In applying the theory to the results of observation, for

any given trajectory, the maximum time of flight is required throughout which the theory is valid. This is given approximately by

$$v_0 e^{-\lambda t} + u_0 = \frac{g}{\lambda} \sin \theta_0 \pm u_0.$$

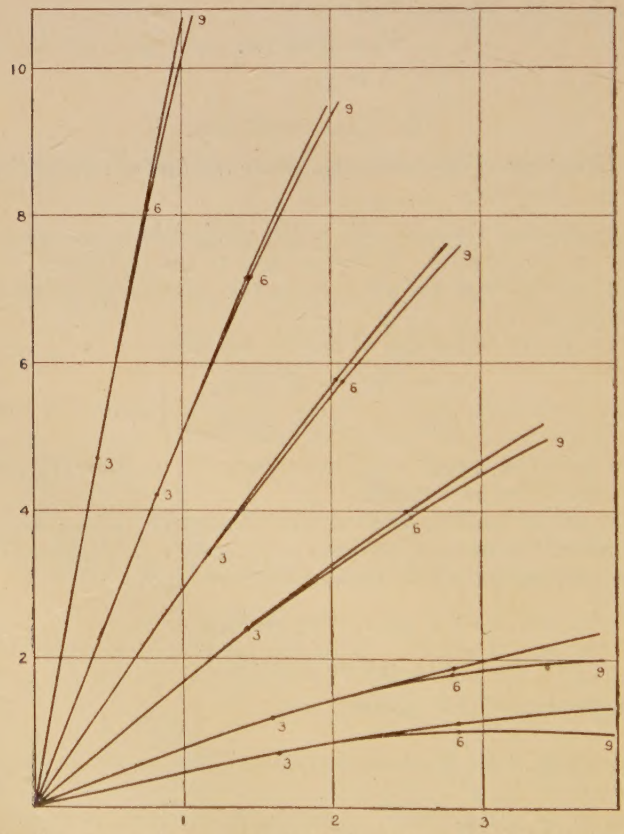
Since u_0 is considerably greater than $\frac{g}{\lambda} \sin \theta_0$, the positive sign on the right hand is alone admissible.

Hence

$$e^{-\lambda t} = \frac{g \sin \theta_0}{\lambda v_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The actual maximum times permissible must be somewhat less than those given by (17).

Fig. 3.



In fig. 3 are shown the initial portions of the trajectories for angles of quadrant elevation 10°, 15°, 30°, 45°, 60°, and 75°.

Horizontal abscissæ represent thousands of yards. Vertical ordinates represent thousands of feet. The upper curve in each case is that calculated by means of equations (16). The lower curve is that obtained by more accurate methods involving a combination of theory and observation. The figures on the curves represent the times of flight.

It may be remarked that, in so far as distance travelled is concerned, the differences between the curves are small. Discrepancies in other respects are to be expected. For, in the first place, the solution is an approximation owing to the assumptions (15). In the second place, no account has been taken, in applying the simple law, of the decrease in the density of the air at high altitudes. It is, however, unexpected that the agreement is so close as it is in the case of the 75° trajectory.

There would seem to be no doubt, however, that, during the first few seconds after the shell leaves the muzzle of the gun, Mallock's law of retardation may be used to study the effect of the diminishing precessional couple upon the oscillations of the shell.

Section 3.—The Solution of the Equations.

In the equation (10) write

$$M_1 + iN = K. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now H is a function of θ only, and is known in terms of the time. Hence the solution of (10) consists of a complementary function and a particular integral.

An approximation to the complementary function was obtained by Horn and Schlesinger (see 'Aerodynamics of a Spinning Shell') by making the substitutions

$$u = e^{i\mathrm{P}z} u_1,$$

$$\mathrm{Q} = e^{i\mathrm{P}z} \mathrm{Q}_1,$$

and treating u_1 and Q_1 as constants.

Making these substitutions it is found that, retaining only the highest power of P,

$$\dot{x}\left(\dot{x}^2 - \dot{x} + \frac{\mathrm{K}}{\mathrm{P}^2}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This is sufficient to show that, with an error of the order 1/P, the solution of (10) may be obtained by omitting Q altogether.

A first approximation to the particular integral is then given by

$$\frac{\partial u}{\partial t} + \frac{\mathrm{K}}{i\mathrm{P}} u = -\mathrm{H},$$

or

$$u\, e^{\frac{\mathrm{K}}{i\mathrm{P}} t} = -\int \mathrm{H}\, e^{\frac{\mathrm{K}}{i\mathrm{P}} t}\, \partial t.$$

If s stand for the stability factor $\frac{\mathrm{P}^2}{4\mathrm{L}}$, the approximation required is

$$u = -\frac{4is\mathrm{H}}{\mathrm{P}}.$$

If $\dot{\theta}$ is taken as constant and $\ddot{\theta}$ neglected, the particular integral in the solution for u is a small quantity of order $1/\mathrm{P}$ which slowly increases. Since $\dot{\theta}$ is always extremely small, over the initial path, compared with P, the particular integral may be neglected throughout.

It is permissible, therefore, to neglect the variation of θ over the initial path. Since F is small, it will be treated as constant.

Equation (10) with the help of (11) reduces to

$$\frac{\partial^2 u}{\partial t^2} - i(\mathrm{P} + i\mathrm{L} + i\mathrm{F})\frac{\partial u}{\partial t} - (\mathrm{M}_1 + i\mathrm{N} + i\mathrm{PF} - \mathrm{LF})u = 0. \quad (20)$$

The solution of this equation is identical with that of a spinning top under a special system of impressed forces.

To solve (20), let

$$u = w\, e^{\left(i\frac{\mathrm{P}}{2} - \frac{\mathrm{L}+\mathrm{F}}{2}\right)t}.$$

Then

$$\frac{\partial^2 w}{\partial t^2} + w\{\mathrm{R}^2 - \mathrm{M}_1\} = 0, \quad . \quad . \quad . \quad . \quad (21)$$

where

$$\mathrm{R} = \frac{\mathrm{P}}{2} + \frac{i(\mathrm{L}-\mathrm{F})}{2} - \frac{i\mathrm{N}}{\mathrm{P}},$$

very approximately. All the terms of highest order in the damping forces L, F, and N have been retained. While real quantities which are small compared with P have been neglected.

It appears in the discussion of the force system that

$$\mathrm{M}_1 = \mathrm{M}\, e^{-\lambda t} \quad \text{approximately,}$$

the approximation again consisting in the neglect of a real quantity which is small compared with P.

Equation (21) can be solved in series, the complete solution being

$$w = A_1 e^{iRt}\left[1 + \frac{M}{m_2^2 + R^2} e^{-\lambda t} + \frac{M^2}{(m_2^2 + R^2)(m_3^2 + R^2)} e^{-2\lambda t} + \dots\right]$$
$$+ B_1 e^{-iRt}\left[1 + \frac{M}{m_2^2 + R^2} e^{-\lambda t} + \dots\dots\dots\dots\dots\dots\right]. \quad \dots \quad (22)$$

The general term of the first series in square brackets may be written in the form

$$\frac{(ix)^n}{\lfloor n (1 + i\mu)(1 + 2i\mu) \dots (1 + ni\mu)}$$

That of the second series in the form

$$\frac{(-1)^n (ix)^n}{\lfloor n (1 - i\mu) \dots\dots\dots (1 - ni\mu)}$$

The moduli of the coefficients of $\frac{x^n}{\lfloor n}$ in these terms are less than unity. Hence each series is less than the exponential series e^x. The series are, therefore, convergent for all finite values of x.

With regard to the periodicity of the solution the following theorem may be proved :—

Let w satisfy the equation

$$\frac{\partial^2 w}{\partial t^2} + wF = 0, \quad \dots\dots \quad (23)$$

where F is a function of t which remains positive throughout the motion. Then w must have an infinite number of real roots. This is true if F is constant but requires formal proof if F is variable. Differentiate (23) n times, then $(n-1)$ times, and so on. Then

$$\left[\frac{D^{n+2}w}{D^n w} + F\right] D^n w + \dots\dots\dots\dots\dots + w D^n F = 0,$$

$$\left[\frac{D^{n+1}w}{D^n w}\right] \quad D^n w + D^{n-1}w \,.\, F + \dots\dots\dots\dots + w D^{n-1} F = 0,$$

$$D^n w + \quad 0 \quad + D^{n-2}w \,.\, F + \dots + w D^{n-2} F = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 \quad + \dots\dots\dots\dots\dots + D^2 w + w \,.\, F = 0.$$

On eliminating $D^n w, \ldots, w$ from these n equations,

$$[D^{n+2}w + D^n w \,.\, F]\, F_1 + D^{n+1}w \,.\, F_2 = 0,$$

where F_1 and F_2 are functions of t.

Since F is always positive, the reality of the roots is established by Fourier's rule.

Now

$$\mu = \frac{\lambda}{2R}$$

$$= \frac{\lambda}{P + i(L - F) - \dfrac{2iN'}{P}}$$

giving as a first approximation to the value of μ

$$\mu = \frac{\lambda}{P}.$$

In practice x never becomes very large, so that a sufficient approximation can be obtained by taking a number of terms of each series for which n is not very large.

In the case of the 3-inch shell $\lambda = \cdot 25$, and $P = 157$, which give for the approximate value of μ:

$$\mu = \frac{\lambda}{P}$$

$$= \cdot 0015. \quad . \quad . \quad , \quad . \quad . \quad . \qquad (24)$$

Consider now the coefficient

$$R_n = \frac{1}{(1 + i\mu) \ldots . (1 + ni\mu)}.$$

$$\log R_n = -\int \partial\mu \left[\frac{i}{1 + i\mu} + \frac{2i}{1 + 2i\mu} + \ldots . + \frac{ni}{1 + ni\mu} \right]$$

$$= -i\Sigma_1^n \tan^{-1}(n\mu) + \log \{(1 + \mu^2) \ldots . (1 + n^2\mu^2)\}^{-\frac{1}{2}}.$$

Hence

$$R_n = \{(1 + \mu^2) \ldots . (1 + n^2\mu^2)\}^{-\frac{1}{2}} \{\cos \Sigma_1^n \tan^{-1}(n\mu) - i \sin \Sigma_1^n \tan^{-1}(n\mu)\}.$$

With regard to x, a first approximation, neglecting terms of the order $1/P^3$, gives

$$x = \frac{M e^{-\lambda t}}{P\lambda}.$$

The coefficients R_n are very involved. It is, therefore, necessary to consider their magnitude.

Since μ is very small the coefficients R_n are nearly equal to unity unless x is so large that a very considerable number of terms of the series requires to be taken. Now very much the most important part of x is $Me^{-\lambda t}/P\lambda$, the other parts being negligible in comparison. At the beginning of the motion the value of x is approximately 12, but after three seconds it drops to about 5.

It is necessary to investigate the nature of the error involved in assuming that each coefficient R_n is equal to unity.

The only term in R_n requiring consideration is $n^2\mu$, such terms as $n\mu$, $n^3\mu^2$, etc., being small in comparison up to the values of n which are required. In R_n, therefore, consider the term

$$-i\mu\{1+2+\ldots.+n\} \quad \text{or} \quad -i\mu\frac{n(n+1)}{2}.$$

The series

$$-i\sum_1^\infty \frac{(ix)^n}{\underline{n}}\frac{n(n+1)}{2}\mu,$$

since such terms as $n\mu$ may be neglected, may be written

$$x\mu - i\sum_2^\infty \frac{(ix)^n}{\underline{n}}\frac{n(n-1)}{2}\mu,$$

or

$$\frac{i\mu x^2}{2}\sum_2^\infty \frac{(ix)^{n-2}}{\underline{n-2}},$$

or

$$\frac{i\mu x^2}{2}e^{ix}. \quad \ldots\ldots\ldots\ldots \quad (25)$$

$\frac{\mu x^2}{2}$ is therefore the error involved, since e^{ix} is the value of the first series when the coefficients R_n are taken as unity. The error involved in the second series is $-\frac{1}{2}\mu x^2$.

In the beginning of the motion this error amounts to about 10 per cent., but after three seconds it drops to about 4 per cent. This is an error in the value of the yaw and the precession of the shell, but, considering the uncertainties of the initial motion of the shell and especially the very small value of the yaw, it is unnecessary to take it into account. It is sufficient to know its approximate magnitude.

It is easy to verify (25). For, writing $w = Ae^{v}$, we have

$$\frac{\partial^2 v}{\partial t^2} + \left(\frac{\partial v}{\partial t}\right)^2 + R^2 - Me^{-\lambda t} = 0. \quad . \quad . \quad . \quad (26)$$

This equation, not being linear, is not adapted to general solution in series, but can be used to obtain an approximate solution.

Let

$$v = a_0 t + a_1 e^{-\lambda t} + a_2 e^{-2\lambda t} + \ldots .$$

On substitution in (26) it is found that

$$a_0 = \pm iR,$$

$$a_1 e^{-\lambda t} = \pm ix,$$

$$a_2 e^{-2\lambda t} = \pm \frac{i\mu x^2}{2}.$$

Hence

$$w = A_1 e^{iRt + ix}\left[1 + \frac{i\mu x^2}{2}\right] + B_1 e^{-iRt - ix}\left[1 - \frac{i\mu x^2}{2}\right] \quad \text{approx.}$$

The coefficients R_n will now be put equal to unity, and the value of x equal to $Me^{-\lambda t}/P\lambda$.

Let

$$\rho = \frac{P}{2}t - \frac{M}{P}u,$$

where

$$u = \frac{1}{\lambda}(1 - e^{-\lambda t}).$$

Let

$$\sigma = -L + \frac{N}{P}$$

and

$$\tau = -F - \frac{N}{P}.$$

Then it can be shown that

$$\phi e^{i\left(\psi - \frac{P}{2}t\right)} = A e^{i(\rho+\alpha)} e^{\sigma t} + B e^{-i(\rho+\beta)} e^{\tau t}, \quad . \quad . \quad (27)$$

where A, B, α, β are real constants to be determined by the initial conditions. On differentiating (27) with respect to t,

$$\left\{\phi + i\phi\left(\dot{\psi} - \frac{P}{2}\right)\right\} e^{i\left(\psi - \frac{P}{2}t\right)}$$

$$= A(\sigma + i\dot{\rho}) e^{i(\rho+\alpha)} e^{\sigma t} + B(\tau - i\dot{\rho}) e^{-i(\rho+\beta)} e^{\tau t}, \quad (28)$$

which can be simplified since σ and τ are extremely small compared with $\dot{\rho}$.

For the initial conditions take a maximum value of ϕ, occurring in the rapid oscillations set up when the shell leaves the muzzle of the gun. In this case $\dot{\phi}$ is initially zero.

Let ψ be measured from the plane of initial maximum yaw.

Then initially,

$$\phi = \phi_0,$$
$$\psi = 0,$$
$$\dot{\phi} = 0,$$
$$\psi = \dot{\psi}_0.$$

Let

$$\frac{\frac{P}{2} - \dot{\psi}_0}{\frac{P}{2} - \frac{M}{P}} = \kappa = \tanh \mu. \quad . \quad . \quad . \quad . \quad (29)$$

These initial conditions are satisfied if

$$\alpha = \beta = 0,$$
$$\phi_0 = A + B,$$
$$-\kappa\phi_0 = A - B.$$

Hence

$$A = (1-\kappa)\frac{\phi_0}{2},$$
$$B = (1+\kappa)\frac{\phi_0}{2}.$$

Then from (27)

$$\tan\left(\psi - \frac{P}{2}t\right) = \frac{M}{N}\tan\rho = K\tan\rho \quad \text{(say)}.$$

The limiting circles between which ϕ oscillates are given by

$$M\frac{\phi_0}{2} \quad \text{and} \quad N\frac{\phi_0}{2},$$

where $M = (1-\kappa)e^{\sigma t} - (1+\kappa)e^{\tau t}$

and $N = (1-\kappa)e^{\sigma t} + (1+\kappa)e^{\tau t}.$

Let

$$-\chi = \frac{\sigma}{2} + \frac{\tau}{2}$$

and

$$-\nu = \frac{\sigma}{2} - \frac{\tau}{2}.$$

Then

$$\left.\begin{aligned} M &= -2e^{-\chi t}\operatorname{sech}\mu\sinh(\nu t+\mu),\\ N &= \ \ 2e^{-\chi t}\operatorname{sech}\mu\cosh(\nu t+\mu). \end{aligned}\right\}$$

These give

$$\tan\left(\psi-\frac{P}{2}t\right)=-\tanh(\nu t+\mu)\tan\rho. \quad . \quad . \quad . \quad (30)$$

These equations represent the whole motion, with

$$\chi=\frac{L+F}{2},$$

$$\nu=\frac{L-F}{2}-\frac{N}{P}.$$

From (30), by differentiation,

$$\dot\psi-\frac{P}{2}=K\dot\rho\,\frac{1+\tan^2\rho}{1+K^2\tan^2\rho}+\dot K\,\frac{\tan\rho}{1+K^2\tan^2\rho}. \quad . \quad (31)$$

With the help of these equations it is possible to discuss very completely the motion of the projectile. It is found that it depends greatly upon the initial value of the precession $\dot\psi_0$.

Case I.—Let $\dot\psi_0=\frac{M}{P}$, *i. e.* let $\kappa=1$, or $\mu=\infty$.

This would be the ordinary slow precession if the forces were constant.

Then $K=1$, nearly, throughout the motion. Also $\dot K=0$. From (31)

$$\dot\psi-\frac{P}{2}=-\rho$$

$$=-\frac{P}{2}+\frac{M}{P}e^{-\chi t}.$$

Hence

$$\dot\psi=\frac{M}{P}e^{-\chi t}.$$

The limiting circles of maximum and minimum yaw are nearly coincident throughout the motion and

$$\phi=\phi_0e^{-\chi t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence the angle of yaw and the rate of precession decrease continually. The motion is completely stable.

Case II.—Consider now the case when $\dot{\psi}_0$ is greater than $\frac{M}{P}$ but less than $\frac{P}{2}$.

Then $\kappa < 1$ and positive.

The initial values of the limiting circles are ϕ_0 and $\kappa\phi_0$.

It is interesting to interpret the motion of the axis of the projectile between its limiting circles.

For this purpose employ equation (31). During a very short interval of time, comprising, however, a number of the very small periods $\frac{\pi}{\dot{\rho}}$, $\dot{K}$ may be treated as sensibly constant.

Initially K is negative and remains always negative.

During the short interval, the beginning of which will be taken as origin of time, $\dot{\rho}$ may be considered constant.

The following sequence is obtained :—

$$\left.\begin{aligned}
&\text{When } t = 0 \qquad \dot{\psi} - \frac{P}{2} = K\dot{\rho},\\
&\quad,, \quad t = \frac{\pi}{4\dot{\rho}} \qquad ,, \quad = \frac{2K\dot{\rho}}{1+K^2} + \frac{\dot{K}}{1+K^2},\\
&\quad,, \quad t = \frac{\pi}{2\dot{\rho}} \qquad ,, \quad = \frac{K\dot{\rho}}{K^2}.\\
&\quad,, \quad t = \frac{3\pi}{4\dot{\rho}} \qquad ,, \quad = \frac{2K\dot{\rho}}{1+K^2} - \frac{\dot{K}}{1+K^2},\\
&\quad,, \quad t = \frac{\pi}{\dot{\rho}} \qquad ,, \quad = K\dot{\rho}.
\end{aligned}\right\}$$

It is seen that the only effect of $\dot{K}$ on the oscillations is to slightly modify the path of the axis from one limiting circle to the other.

During the period $\frac{\pi}{\dot{\rho}}$,

$$\tan\left(\psi - \frac{P}{2}t\right) = K \tan \dot{\rho}t.$$

By the end of this period ψ has increased by an amount

$$\begin{aligned}
\psi_1 &= \frac{P}{2}\frac{\pi}{\dot{\rho}} - \pi\\
&= 2\pi M e^{-\lambda t} \quad \text{approx.}
\end{aligned}$$

Hence

$$\dot{\psi}_1 = \frac{M}{P} e^{-\lambda t}, \quad \text{approximately.}$$

If C be the point of contact of the path of the axis with the outer circle, say, then $\dot{\psi}_1$ is the velocity of this point of contact.

It is not difficult, now, to understand the course of the projectile's axis. It is probable that, at the instant of leaving the muzzle of the gun, the motion of the projectile is extremely unsteady. But, at a very short distance from the muzzle, the projectile settles down to a stable gyroscopic motion, with its axis nearly tangential to the trajectory. The case of $K=1$, which corresponds to $\kappa=1$, has been discussed.

If $\kappa<1$, K is initially less than 1 and negative. $\dot{\psi}$ then becomes zero in the passage of the axis from the outer to the inner circle, and again on the passage back to the outer circle.

Hence there are loops in the path, the nodes occurring near the outer circle. As K increases in absolute value with time, the nodes in the path approach the inner circle.

As K approaches the value -1 with increase in t, $\dot{\psi}$ settles down to a slow precessional value.

Case III.—If κ and therefore μ are negative, but $|\kappa|<1$, then K is initially positive. In this case $\dot{\psi}$ may not become zero during several of the periods $\pi/\dot{\rho}$, in which case the nose of the projectile precesses right round its direction of motion with a rapid precession. But with increase of time K becomes negative and the course of the axis follows that described in Case II. Here also $\dot{\psi}$ settles down to the slow precession. The above theory, upon comparison with experiment, shows definitely that ν is positive.

Consider now the theory of the limiting circles of yaw. With increase of time they approach one another and take the common value

$$e^{-(\lambda-\nu)t}=e^{\left(F+\frac{N}{P}\right)t}.$$

Hence if $F+\frac{N}{P}$ is positive the motion is always stable.

If $\chi=0$ it is seen, from (32), that if the shell start off with the proper precession the yaw remains constant. But for any other value of the initial precession the motion is unstable.

If χ is zero and β negative while ν is positive, both circles decrease for a time $t=\frac{\mu}{\nu}$. At this time the inner

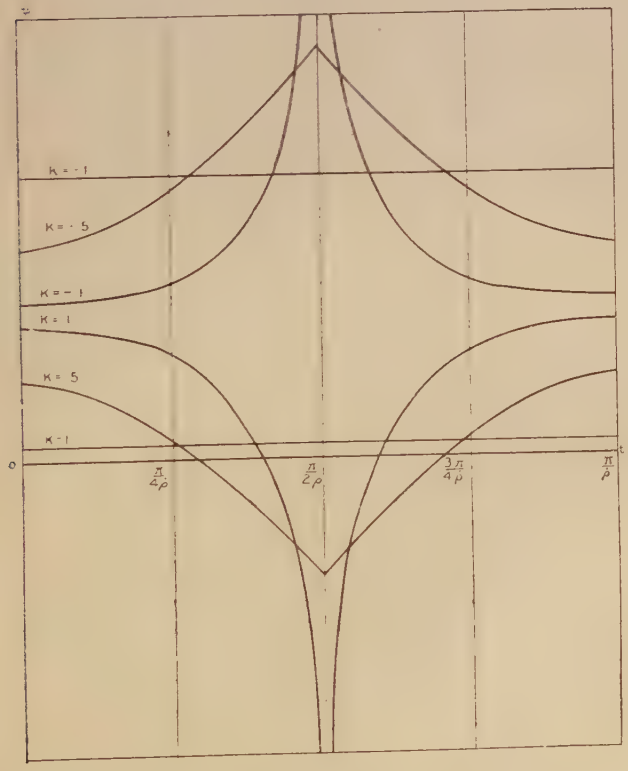
circle becomes zero and the outer a minimum. Thereafter both circles increase.

A motion in which the yaw initially decreases does not necessarily imply that the motion is subsequently stable.

The coefficient L always has a damping and therefore stabilising effect. F is shown by the theory and observations of drift to be positive, and therefore also has a stabilising effect.

In fig. 4 is shown the variation of $\dot{\psi}$ for different values

Fig. 4.



of K. The figure is not intended to be exactly to scale, and for simplicity 2ρ has been put equal to P−·1P, *i. e.*, to ·9P. The values of K taken are

$$1, \quad \cdot5, \quad \cdot1, \quad -\cdot1, \quad -\cdot5, \quad -1.$$

For each value of K, $\dot{\psi}$ is plotted throughout the period $\frac{\pi}{\rho}$,

which is the time of passage of the axis of the projectile from the outer circle to the inner and back to the outer. K is assumed to be constant throughout this small interval.

With regard to the particular integral, it is quite negligible till $\dot{\theta}$ becomes large, but it then becomes important for both s (the stability factor) and $\dot{\theta}$ are then large. It has been suggested that some means should be adopted to decrease the spin more rapidly and thus reduce s, so as to lessen this disturbing factor at the apex of a trajectory.

For initial elevations up to as much as 45° or 50° the effect of the trajectory on the yaw is negligible. The explanation of this important fact has only recently been given and is worthy of a simple explanation in words.

We have the well-known results in the theory of the spinning-top: if the precession is hurried the top rises: if it is retarded the top sinks. Now the projectile, neglecting the motion of its centre of gravity, is like a top spinning upon its point. The centre of gravity of the projectile corresponds to the point of the top. The air-pressure acting through the centre of pressure corresponds to gravity acting upon the top. In the case of the top, if the gravity couple were increased, the precession would be hurried and the top would rise. Now the curvature of the path, when the projectile's axis is pointing above the tangent, tends to increase the pressure-couple. Hence the precession is hurried and the nose of the projectile is lowered. When its axis is pointing below the tangent, the curvature of the path tends to decrease the pressure-couple. Hence the precession is retarded and the nose of the projectile is lowered still further.

It will be noted that the oscillations of the axis of the projectile are referred to the direction of motion of the centre of gravity of the projectile. There should, therefore, be added the helical oscillations of the latter in order to obtain the resultant oscillations relatively to the plane trajectory. It has, however, been shown that the angular helical oscillations of the centre of gravity are, so far as their most important periodic term is concerned, only a very small fraction of ϕ, the yaw of the shell referred to the direction of motion of the centre of gravity. It is, accordingly, quite unnecessary to take them into account in questions connected with the stability of the projectile.

CXI. *An Experimental Determination of the Inertia of a Sphere Vibrating in a Liquid.* *By* N. C. KRISHNAIYAR, *M.A., Lecturer in Physics, University College, Rangoon* *.

SIR G. G. STOKES † has shown analytically that in the case of a sphere attached to a fine wire of which the effect is neglected, and swung in an unconfined mass of incompressible liquid, the force exerted on the sphere by the motion of the surrounding liquid is given by the relation

$$F = -A\frac{d^2y}{dt^2} - B\frac{dy}{dt},$$

where y is the displacement and t is the time and the coefficients A and B have the following values,

$$A = (\tfrac{1}{2} + 9.4\lambda r)M' ; \quad B = (9.4\lambda r)\left(1 + \frac{1}{\lambda r}\right)M'a,$$

where M′ is the mass of the fluid displaced by the sphere, r is the radius, $a = 2\pi/T$ where T is the period of oscillation, and $\lambda = (a/2\mu')^{1/2}$ and μ' = coefficient of viscosity ÷ density. The first term in the expression for the force F has the same effect as increasing the inertia of the sphere. To take this term into account it will be sufficient to conceive a mass M collected at the centre of the sphere, adding to its inertia without adding to its weight. The second term gives a frictional force varying as the velocity, and the effect is mainly to produce diminution in the arc of oscillation. The assumptions made were that the centre of the sphere performed small periodic oscillations along a straight line, that the sphere had a motion of translation only, that the velocities were so small that their squares could be neglected, and that there was no slipping of the fluid along the surface of the solid in contact with it. The application of the theory to the pendulum observations of Bessel and Baily was very satisfactory in the case of pendulums swinging in the air; but in the case of pendulums swinging in water or a more viscous liquid, conclusive results were not possible on account of the rapid diminution of the arc of oscillation.

G. F. McEwen ‡ measured the force acting on a sphere swinging in water and in a very viscous oil by employing a forced vibration method in which the sphere was suspended

* Communicated by the Author.

† 'Cambridge Philosophical Transactions,' 1850; 'Collected Papers,' vol. iii. p. 34. The same result has been obtained by a different method. Lamb's 'Hydrodynamics,' 3rd ed., p. 584.

‡ Phys. Rev., 5th ser. vol. xxxiii. pp. 492–511 (1911).

by a fine wire, and had a slow motion of translation only along a vertical line. The oscillations were small, and either diminished very slowly or were maintained constant for any desired length of time. The forcing was done by a mechanical arrangement and a satisfactory measurement of the force was obtained. H. T. Barnes*, by an arrangement in which a spherical pendulum bob moving under water was received by a ballistic pendulum, found that its effective inertia comes from the addition to its own mass of half the mass of the liquid that would fill the space occupied by the sphere. Gilbert Cook† determined the inertia of a large spherical mine-case falling freely through water under the influence of gravity. The theoretical increase of inertia in such cases is $\frac{1}{2}$ the mass of the displaced liquid, and Gilbert Cook obtained the value 0·46.

In the present paper is described an arrangement in which the inertia of bodies vibrating with constant frequency inside a liquid can be determined under conditions which approximate, to a large extent, to the assumptions made in the hydrodynamical theory. The author has found that thin wires can be maintained in steady transverse vibration if they carry a small direct current and are placed between the poles of an electromagnet actuated by a single-phase alternating current, if the length and tension of the wires are adjusted to give the same natural frequency as the number of cycles of the alternating current. With the same current and the same length of wire the range of tension for the vibration to be maintained is limited, and the tension for resonance can be found from that which gives the maximum amplitude. If a small mass m be at the middle point of a wire of length l and mass per unit length σ, and if m be large in comparison with σl, the tension required to give the system a natural frequency n is given by the relation ‡

$$\mathrm{T}=\pi^2 n^2 l m\left(1+\frac{\sigma l}{6m}\right)^2.$$

If the tension is applied by hanging a mass M from the free end of the wire passed over a light pulley, $\mathrm{T}=\mathrm{M}g$. If T_1 and T_2 be the values of the tension for best resonance, determined experimentally, when the system is in air and when it is immersed in a large mass of a liquid, and if m_1

* 'Transactions of the Royal Society of Canada,' vol. xi. June Sept. 1917.

† Phil. Mag. vol. xxxix. (1920).

‡ Rayleigh's 'Sound,' vol. i. art. 136.

and $m_1 + km_2$ be the inertia of the body in air and in the liquid,

$$\frac{T_1}{T_2} = \frac{m_1 + km_2}{m_1} \left\{ \frac{1 + \frac{(\sigma l)}{6(m_1 + km_2)}}{1 + \frac{\sigma l}{6m_1}} \right\}^2,$$

where m_2 is the mass of the liquid displaced by the body and k is the inertia factor of the body in the liquid. The wire also increases in inertia when in the liquid. In the experiments the mass of the wire σl is small in comparison with the mass of the body, and the increase is allowed for by making use of Stokes's table * for cylinders vibrating transversely. Moreover, the quantities σl and m_1 were so chosen that when Stokes's theory was applied the quantity within the bracket was very nearly equal to unity. The frequency n of the maintained vibration does not enter the final formula, and all that is necessary is the frequency of the alternating current should be steady during the experiment.

A silver-plated steel wire of radius 0·009 cm., fixed at one end, passed horizontally under two sharp wedges 30 cm. apart, and then over a fixed light aluminium pulley with ball bearings. Weights could be added to a pan attached to to the free end of the wire. The fixed point and the top of the pulley were slightly higher than the level of the edges, so that the wire was well pressed against the edges. Spheres were fixed with gelatine-glycerine cement in the middle of the wire between the wedges. Near the two edges, on either side of the wire parallel to its length, were the pole-pieces, 10 cm. long, 3 cm. wide, 1 cm. high, of two similar electromagnets. Hence the first one-third and the last one-third parts of the wire were in a periodic field of force, while the middle one-third part containing the sphere was unrestricted in space. The apparatus was placed inside a tank with glass sides, 3 ft. long, $1\frac{1}{2}$ ft. broad, and 2 ft. high. A small direct current about one-third of an ampere in the wire and an alternating current of about two amperes through the magnets maintained a steady vibration of the wire in air in the vertical plane if its tension was adjusted for resonance. Kerosene oil (viscosity $\mu = 1{\cdot}67 \times 0{\cdot}0088$) was poured into the tank, the currents were increased to about half an ampere and four amperes respectively, and the tension to produce resonant vibration was determined. Amplitude of the vibration was observed with a telemicroscope having a glass scale

* 'Collected Papers,' vol. iii. p. 52.

in the eyepiece. Correct tension could be determined within one per cent. in air and two per cent. in oil. A preliminary experiment with the wire alone without the sphere showed general agreement with Stokes's tabulated value for thin cylinders.

Three determinations were made: first with a pith ball coated with lead foil and then with gelatine-glycerine cement to make it impervious to kerosene oil, second with another pith ball covered with aluminium paint and then with the gelatine-glycerine cement, and third with a polished hollow silver ball. Gelatine melted with a little glycerine produced a cement insoluble in kerosene and was therefore used for fixing the balls on the wire.

	1st Ball.	2nd Ball.	3rd Ball.
Mass of ball in air	2·1307	1·1847	1·5035 gram.
Mass of oil displaced by ball	1·8614	1·1172	2·1323 gram.
Diameter of ball	1·7	1·36	1·7 cm.
Diameter of wire	0·0182	0·0182	0·0235 cm.
Wt. in air of 30 cm. of wire.....	0·0584	0·0584	0·1011 gram. wt.
Wt. of oil displaced by wire	0·0059	0·0059	0·01 gram. wt.
Best value for tension in air.....	1530	780	1010 gram. wt.
Best value for tension in oil	2310	1210	1840 gram. wt.
The value of k obtained by Exp....	·584	·585	·580
The value of k by Stokes's Theory.	·530	·536	·530

The excess of the experimental results over the theoretical ones, though small, is larger than the possible errors of experiment. In all the three cases the quantity

$$\{1 + \sigma l_1^6 (m_1 + k m_2) \div 1 + \sigma l^6 m_1\}^{\frac{1}{2}}$$

was equal to 1 correct to two places of decimals, and therefore the approximations used therein could not sensibly affect the calculation. The alternating current was taken from the town supply, and in answer to a query the Chief Electrical Engineer of the Electric Supply Company wrote: "The frequency is kept constant at all times at 50 periods. There is a remote possibility of a very slight variation for a few seconds while turbines are being changed over. This would not amount to more than 1 per cent. at the outside. As the turbines are changed over not more than two or three times in the 24 hours, any slight variation would be only momentary." As has been pointed out before, the actual value of the frequency does not matter if it is constant during a determination. The wire when inside the liquid was not sensibly heated. As is required by the hydrodynamical theory, the centre of the sphere performed *small*

periodic vibrations in a straight line without any rotation, but possibly the velocities were not so small that their squares could be neglected. Gilbert Cook * has found experimentally that the frictional resistance to the gravitational fall of a sphere inside water, being mainly due to skin friction and eddy formation, is proportional to the square of the velocity. The spheres were about 7 cm. below the liquid surface and about 15 cm. above the bottom of the tank. The magnets and armatures were 5 cm. on either side. The maintained oscillation in air was relatively of much larger amplitude than in the liquid. The effect of large amplitude in air, the restriction of space due to the magnets in the neighbourhood, and the increased frictional resistance due to the square law will all tend to approximate the experimental and the theoretical values.

University College, Rangoon.
May 20, 1923.

Postscriptum dated Oct. 4, 1923.—Prof. Barnes's paper cited above, a copy of which was obtained only a few days ago, contains a footnote that Chevalier Du Buat in Volume 2 of the second edition of his 'Principes d'Hydraulique,' Paris, 1786, had already announced what was practically Stokes's law, and that Du Buat's experiments gave ·585 for the coefficient of the inertia increase due to the fluid carried by the sphere.

CXII. *Electronic Conduction in Metals.* By Arthur Bramley, *Fellow in Physics, Princeton University* †.

THE theory of metallic conduction owes its foundation primarily to the researches of Drude and Sir J. J. Thomson. The former introduced the conception of the free electron into the theory: this theory, however, encountered unsurmountable difficulties in the phenomena observed by Kamerlingh Onnes at low temperatures, and has gradually been superseded by the doublet theory due to J. J. Thomson, according to which the electric current is due to the influence of the external field on the orientation of the doublets. The object of this discussion is the derivation of (and expression for) the law of force, operative near the nucleus, and the explanation of the doublet structure assumed in that theory.

* Phil. Mag. vol. xxxix. (1920).
† Communicated by Prof. E. P. Adams.

We shall first calculate the magnetic and electric forces at any point in space due to the revolution of a single electron in a circular orbit about a positive nucleus with constant angular velocity.

In order to carry out the calculation, it is necessary, following the method initiated by Kirchhoff, to consider the types of solution of the differential equation for the potential.

The potentials are defined in the equations :

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho,$$

and

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = -\rho V/c.$$

The intensities are obtained by differentiation :

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \text{grad}\,\phi,$$

$$H = \text{rot}\,U.$$

As a result of this calculation, we find

$$E_R = \frac{e}{4\pi}\frac{1}{\sqrt{1-\beta^2}}\cdot\frac{r-R\cos\theta_1}{(r^2+R^2+z^2-2rR\cos\theta_1)^{3/2}},$$

$$E_z = \frac{e}{4\pi}\frac{1}{\sqrt{1-\beta^2}}\cdot\frac{z}{(r^2+R^2+z^2-2rR\cos\theta_1)^{3/2}},$$

$$E_\theta = \frac{e}{4\pi}\frac{1+\beta^2}{1-\beta^2}\cdot\frac{R\sin\theta_1}{(r^2+R^2+z^2-2rR\cos\theta_1)^{3/2}},$$

$$H_\theta = 0,$$

$$H_z = \frac{e}{4\pi}\frac{\beta}{\sqrt{1-\beta^2}}\cdot\frac{r-R\cos\theta_1}{(r^2+R^2-2rR\cos\theta_1)^{3/2}},$$

$$H_R = \frac{e}{4\pi}\frac{\beta}{\sqrt{1-\beta^2}}\cdot\frac{-z}{(r^2+R^2+z^2-2rR\cos\theta_1)^{3/2}},$$

where $(r^2+R^2+z^2-2rR\cos\theta_1)^{1/2}$
= distance from centre of electron to point in question for intensity,

R = radius of electronic orbit,

θ_1 = angle between radius vector to electron and projection of $(r^2+R^2+z^2-2rR\cos\theta_1)^{1/2}$ on orbital plane,

β = velocity / c.

The field obtained in this case corresponds to the field produced by the simplest Bohr atom—hydrogen—when the single electron is in a non-radiating state: *i. e.*, is revolving about the nucleus in one of the quantum rings.

On account of the non-symmetrical distribution of charge, the electric and magnetic intensities at any point vary with the time, being periodic functions with period equal to that of the revolving electron *.

Geometrical Considerations.

The most fundamental property of space is that its points form a three-dimensional manifold. That is, the relationship of any given point with respect to the other point can be expressed by means of three quantities—the coordinates of the new point with respect to the old. Another familiar example of a three-dimensional manifold is the Gibbs's thermodynamic diagram, in which the properties of substances are represented in a three-dimensional manifold whose coordinates of reference are volume, entropy, and energy. The adiabatic expansion of a gas is represented by paths, straight lines in this case, parallel to the volume axis. The possible positions of a rigid body can be represented by points in a six-dimensional manifold, while the possible configurations of a system of n degrees of freedom constitute, in general, an n-dimensional manifold. The characteristic of an n-dimensional manifold is that each of the elements composing it may be defined by giving n quantities—the coordinates.

If we take any two points in a three-dimensional manifold, $P(x\,y\,z)$ and $Q(x_1\,y_1\,z_1)$, then the distance between these two points is, by the fundamental proposition of Euclidean geometry (Pythagorean Theorem),

$$s^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 ;$$

or, if we take these points infinitely near to one another, so that

$$P = P(x\,y\,z)$$

and $$Q = Q(x+dx,\ y+dy,\ z+dz),$$

then this distance becomes

$$ds^2 = dx^2 + dy^2 + dz^2.$$

* Bramley, Phil. Mag. October 1922, p. 720.

Similarly, for an n-dimensional Euclidean geometry, the differential distance between $P(x_1 \ldots x_n)$ and $Q(x_1+dx_1 \ldots)$ is

$$ds^2 = dx_1^2 + dx_2^2 \ldots dx_n^2.$$

The natural extension of this idea, which was first pointed out by Riemann, is to consider the case where the fundamental form, as it is called, may be expressed in the form

$$ds^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 \ldots g_{nn}dx_n^2,$$

where ds^2 measures the infinitesimal distance between two neighbouring points in a more general manifold than that conceived by Euclid.

The distance between two given points, $P(x_1 \ldots x_n)$ and $Q(x_1 \ldots x_n)$, measured along a definite curve, will be

$$s = \int ds = \int \sqrt{\Sigma g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \, . \, dt,$$

where the path of integration is along the simple curve connecting these two points.

This idea of gaining knowledge of the external world through a consideration of its infinitesimal parts is one which transcends physics as well as mathematics. From a knowledge of the way in which the velocity of a particle changes on the infinitesimal length of a path, we can find the time taken to traverse a certain distance along that path; so, in Riemannian geometry, from a knowledge of the distance between every two infinitely near points, we can calculate the distance between points a finite distance apart. This principle is the basis of Differential Geometry.

A difficulty may have been noticed in this presentation with respect to the measure of distance. Suppose, for example, that we have to find the distance between two points on the x_1 axis, $x_1=0$ and $x_1=10$, say, in a manifold in which

$$g_{11} = x_1^2.$$

Then

$$s = \int_0^{10} \sqrt{x_1^2 \left(\frac{dx_1}{dt}\right)^2} \, . \, dt = \int_0^{10} \, . \, x_1 dx_1 = 50.$$

The distance between them is 50, whereas we have only measured the length along the x_1 axis from $x_1=0$ to $x_1=10$.

The solution is obvious. The system of coordinates allows us only to label each point—*i. e.*, when we are given the set of numbers $x_1=10$, $x_2=x_3 \ldots =0$, we are able to pick out a definite point in space which is characterized by this label. But to return to the measure of distance: if we wish to find

the distance between $x_1=0$ and $x_1=10$ along the x_1 axis, we take a measuring rod (our unit) and, starting from $x_1=0$, measure on this unit the length between $x_1=0$ and $x_1=dx_1$, and so on, until we come to $x_1=10$; then the sum of all these measurements gives us the length between $x_1=0$ and $x_1=10$ measured along the x_1 axis with one definite unit rod. In a Euclidean manifold, we first choose the coordinate axis and then choose the points on them corresponding to $x_1=1$, $x_1=2$, ..., etc., so that the distance between $x_1=1$ and $x_1=0$ etc. is unity when measured on the definite rod we are given.

If we change our rod, then this simple relation will no longer hold, provided the new rod is not identical with the old. A manifold is thus completely defined, in the Riemannian sense, when we have labelled every point with reference to a given coordinate system, and, further, have been given a unit of measurement. Riemann defined his unit of measurement by means of the equation

$$ds^2 = \sum_{ij} g_{ij} dx^i dx^j.$$

Now this system is not yet completely defined; the ambiguity still existing was first pointed out by Weyl. Suppose we wish to compare the distance between a set of points at A and another set at B. In order to do this, we must first measure the distance between the points at A on our measuring rod, next carry it from A to B, and then measure the distance between the points at B. But how do we know that we will get the same value for this comparison if we move the measuring rod from A to B by different paths? We don't. Thus, in order to measure this ratio, we must not only fix our standard of measurement, but also the path along which this measure is transported between the given points. In general, in order completely to define our manifold, we must determine first a coordinate system through which we can identify the points in the manifold, and next place at each point in this manifold a measuring rod which is associated with this point only. Then, if we choose any path between two points A and B, the length along any curve connecting these points will be the sum of the measurements at all points P of this path, of the distance between $P(x_1 \ldots x_n)$ and $P(x_1+dx_1 \ldots dx_n)$ measured on the rod at the point P.

If the manifold is calibrated in a definite manner by giving at each a definite measuring rod, and if we take the

measuring standard at any point $P(x_1 \dots x_n)$ and move it to a neighbouring point $P(x_1+dx \dots)$, then there will be a relation between the length of the standard associated with the point P as measured by the standard at P and its original length as measured by itself; or, if dl is the change in length,

$$\frac{dl}{l} = -d\phi(x_1 \dots x_n).$$

If we now associate with each point a new set of standards, such that $\bar{l}$, the length of the new standard measured in terms of the old standard, satisfies the equation

$$\bar{l} = \lambda l, \quad \lambda = \lambda(x_1 \dots x_n),$$

then, as before,

$$\frac{d\bar{l}}{\bar{l}} = -d\phi$$

or

$$d\bar{\phi} = d\phi - d\log\lambda.$$

In order that we can find a λ such that $d\phi=0$,

$$d\phi = d\log\lambda \text{ (a differential form)}$$

or

$$d\phi = \sum_i \psi_i dx^i, \quad \psi_i = \psi_i(x_1 \dots x_n).$$

Thus the metrical character of a manifold is characterized relatively to a system of reference by two fundamental forms, namely a quadratic differential form,

$$ds^2 = \sum g_{ij} dx^i dx^j,$$

and a linear one,

$$d\phi = \sum_i \psi_i dx^i.$$

If the calibration is changed, ψ_i and g_{ij} must be replaced by

$$\psi_i - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x^i} \quad \text{and} \quad \lambda g_{ij}.$$

In four-space (x, y, z, t),

$$\frac{dl}{l} = \psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3 + \psi_4 dx_4.$$

Integrating,

$$\log l/l_0 = \int (\psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3 + \psi_4 dx_4),$$

and the length l will be independent of the path, provided

$\psi_1 dx_1 + \psi_2 dx_2 + \psi_3 dx_3 + \psi_4 dx_4$ is a perfect differential. The condition for this is

$$\frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} = 0, \quad \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} = 0, \quad \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} = 0,$$

$$-\frac{\partial \psi_4}{\partial x} - \frac{\partial \psi_1}{\partial t} = 0, \quad -\frac{\partial \psi_4}{\partial y} - \frac{\partial \psi_2}{\partial t} = 0, \quad -\frac{\partial \psi_4}{\partial z} - \frac{\partial \psi_3}{\partial t} = 0.$$

If ψ_1, ψ_2, ψ_3, and ψ_4 were the potentials of the electromagnetic field, these equations would give us precisely the three components of the electric and magnetic forces which are zero in this case.

Natural phenomena must obviously be independent of the particular coordinate system we use, or the particular standards of length at each point in the manifold. Accordingly, all quantities or relations that represent these conditions analytically must contain the functions g_{ij} and ψ_i in such a way that invariance holds:

(i.) For any transformation of coordinates (coordinate invariance);

(ii.) For the substitutions which replace g_{ij} and ψ_2 by λg_{ij} and $\psi_2 - \frac{\partial}{\partial x_2} \log \lambda$ respectively, no matter what function of the coordinate λ may be (calibration invariance).

When we are given the g's and ψ's, the calibration and coordinate invariance enables us to determine all the physical properties which are represented by this manifold.

In order to determine the g's, we shall assume that our manifold is of such a character that the trajectories of the particles in which we are interested for our special problem (mechanical) will be geodesics in the manifold. This assumption, which is tacitly assumed by Einstein and other workers in this field, really amounts to a device for integrating the differential equations of motion of the particle. The geodetic path along which a particle moves under the action of no forces will be given by the equation

$$g_{ik} \frac{d^2 x^i}{ds^2} + \Gamma_{rsk} \frac{dx^r}{ds} \frac{dx^s}{ds} = 0,$$

where

$$ds^2 = \sum_{ik} g_{ik} dx^i dx^k$$

and

$$\Gamma_{rsk} = \frac{1}{2} \left(\frac{\partial g_{rk}}{\partial x_s} + \frac{\partial g_{sk}}{\partial x^r} + \frac{\partial g_{rs}}{\partial x^k} \right).$$

If we define

$$\mathrm{H} = \sum_{ik} \rho_0 g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = \rho_0,$$

then our equation may be written

$$\frac{d}{ds}\left(\frac{\partial \mathrm{H}}{\partial \dot{x}^r}\right) - \frac{\partial \mathrm{H}}{\partial x^r} = 0.$$

If we are given a problem of a particle under the action of no forces, whose kinetic energy

$$\mathrm{T} = \sum_{i} \rho_0 \left(\frac{dx}{ds}\right)^2,$$

then from a comparison with the above equation we see that

$$g_{ij} = \left.\begin{matrix} 0 \\ \rho_0 \end{matrix}\right\} \begin{matrix} i \neq j \\ i = j, \end{matrix}$$

or the manifold is Euclidean.

When the particle is under the action of no forces, the problem has as yet not in general been solved, although Schwarzschild has found a solution for the case of a particle under the action of the gravitational field of a spherical body.

For this case, in polar coordinates,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2,$$

where m = mass of attracting body.

We shall show, in the sequel, how the potential ψ of the electric field corresponding to a positive nucleus can be found from a knowledge of the g's, subject to the above restrictions.

To test our new hypothesis—the invariant character of physical laws—we must choose an initial invariant law, and work out its implication. We shall choose as our initial law a Hamiltonian principle—that the change in our fundamental integral invariant $\int \mathrm{H}\, dx$ is zero for every small variation of the structure of the manifold.

In the case of a change of coordinates, the integral must be expressed in invariant form, *i. e.* we must express the law in the following way:

$$\delta \int \mathrm{H} \sqrt{-g}\, dx_1 \ldots dx_4 = \delta \int \mathfrak{H} dx = 0,$$

where

$$\mathfrak{H} = \mathrm{H} \sqrt{-g}.$$

Since every Riemannian space possesses an integral of

this form *, the laws of perfect fluids which follow as a natural consequence of its invariance must be valid in all Riemannian spaces.

In the case of a change in the calibration of the universe, the integral must be expressed in slightly different form, namely:

$$\delta \int \mathrm{H}\, dx = 0.$$

Since the density ρ_0 is of weight zero (see sequel, p. 1067), the tensor density H is a calibration invariant, and thus we are led to the above form as the correct expression of the Hamiltonian law, in the case of electromagnetic phenomena.

First consider the case where we subject the manifold to a continuous deformation, so that the point x_i becomes transformed into the point

$$x_i = x_i + \xi(x_1 \ldots x_n) \cdot \epsilon,$$

where ϵ is an infinitesimal parameter (Lie's transformation). Then

$$\delta \int \mathbf{H}\, dx = \int \frac{\partial (\mathbf{H}\xi^i)}{\partial x^i}\, dx + \int \delta \mathbf{H} \cdot dx,$$

where the first integral represents the change in **H** due to a change in the contour of the region of integration (*i.e.*, a change in the limits of integration).

Now, since we can write

$$-\delta \mathbf{H} = \mathbf{H}_i^k \frac{\partial \xi^i}{\partial x^k} + \frac{1}{2} \frac{dg_{rs}}{dx_i} \cdot \mathbf{H}^{rs} \xi^i \quad \text{(summed)} :$$

defining

$$\mathrm{V}_i^k = \mathbf{H}_i^k - \delta_i^k \cdot \mathbf{H},$$

where

$$\delta_i^k = \begin{matrix} 0 & i \neq k \\ 1 & i = k \end{matrix} \Big\},$$

the equation becomes

$$-\delta \int \mathbf{H}\, dx = \int \frac{\partial (\mathrm{V}_i^k \xi^i)}{\partial x^k}\, dx + \int \left\{ -\frac{\partial \mathbf{H}_i^k}{\partial x^i} - \frac{1}{2} \frac{\partial g_{rs}}{\partial x^i} \cdot \mathbf{H}^{rs} \right\} \xi^i\, dx.$$

Therefore

$$\frac{\partial (\mathrm{V}_i^k \xi^i)}{\partial x^k} = \mathrm{V}_i^k \frac{\partial \xi^i}{\partial x^k} + \frac{\partial \mathrm{V}_i^k}{\partial x^k} \xi^k = 0 ;$$

i.e., $$\mathrm{V}_i^k = 0, \quad \text{or} \quad \delta_i^k \cdot \mathrm{H} = \mathrm{H}_i^k,$$

and also

$$\mathrm{H}\ {}_{ik}^{k} = \frac{\partial \mathbf{H}_i^k}{\partial x^k} + \tfrac{1}{2} \mathbf{H}^{rs} \frac{\partial g_{rs}}{\partial x_i} = 0.$$

* L. P. Eisenhart, Proc. Nat. Acad. January 1923, p. 4.

Now the equation above is equivalent to the general equation of a fluid under the action of forces.

Assuming Euclidean coordinates, *i. e.* a fluid under the action of no forces, the equation can be written

$$\frac{\partial \mathrm{H}_i^k}{\partial x^k} = 0.$$

The velocity $u=\frac{dx}{ds}$ etc. may be written as two parts, the first referring to the motion u_0 etc. of the centre of mass of the particle in an element, the second to the interior motion u_1 etc. relative to the centre of mass. With regard to the last part, $\Sigma \rho u_1 v_1$, for instance, represents the rate of transfer of u-momentum across unit area parallel to the y-plane, and is therefore equivalent to the stress usually denoted by p_{xy}.

Taking $i=4$, we get the well-known equation of continuity:

$$\frac{\partial(\rho u_0)}{\partial x}+\frac{\partial(\rho_0 v)}{\partial y}+\frac{\partial(\rho_0 w)}{\partial z}+\frac{\partial \rho}{\partial t}=0.$$

Taking $i=1$,

$$-\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)=\rho\frac{\partial u_0}{\partial t},$$

where we have taken

$$g_{ik}=\begin{matrix}0\\1\end{matrix}\Big\}\begin{matrix}i\neq k\\ i=k.\end{matrix}$$

Accordingly, the whole of the dynamics of material systems is contained in the integral invariant equation; further, from the consideration of calibration invariant conditions associated with the Hamiltonian integral, we shall find that the electromagnetic laws of the field are a consequence of the invariant nature of the equations. The prediction of these well-known properties of matter seems to be a valuable confirmation of the theory, aside from its intrinsic value as an explanation of gravitational forces, and further, as we shall see, of the interatomic dynamics.

We have here to add the further restriction that the Hamilton Principle must also be invariant with respect to re-calibration, in which ψ_i and g_{ik} are replaced respectively by $\psi_i-\frac{\partial}{\partial x_i}\log\lambda$ and λg_{ik}, where λ is an arbitrary function of position (positive), or if the calibration ratio between the altered and the original calibration is $\lambda=1+\pi$,

where π is an infinitesimal scalar. As a result of this process, the fundamental quantities assume the following increments:

$$\delta g_{ik} = \pi g_{ik} \quad \text{and} \quad \delta \psi_i = -\frac{\partial \pi}{\partial x_i}.$$

Since the variation of the action must vanish,

$$\delta \int \mathrm{H}\, dx = \int \frac{\partial}{\partial x^k}(\delta \nu^k)\, dx + \int (\mathrm{W}^i \delta \psi_i + \tfrac{1}{2}\mathrm{W}^{ik} \delta_{(ik)})\, dx = 0,$$

where $\mathrm{W}^i = \dfrac{\partial \mathrm{H}}{\partial \psi^i}$ and $\delta \nu^k$ is defined uniquely by the above equation.

Substituting in the equation the above values of $\delta_{(ik)}$ and $\delta \psi_i$, and further setting

$$\delta \nu^k = \pi \cdot s^k + \frac{\partial \pi}{\partial x^r} \cdot h^{kr}.$$

Then

$$\delta \int \mathrm{H}\, dx = \int \frac{\partial^2 \pi}{\partial x^r\, dx^s} h^{rs} + \int \frac{\partial \pi}{\partial x^k}\left(s^k + \frac{\partial h^{rk}}{\partial x^r} - \mathrm{W}^k\right) dx + \int \pi \left(\frac{ds^k}{dx^k} + \tfrac{1}{2}\mathrm{W}_i^i\right) dx,$$

from which we have at once

$$s^k + \frac{\partial h^{rk}}{\partial x^k} = \mathrm{W}^k,$$

$$h^{kr} + h^{rk} = 0,$$

$$\frac{\partial s^k}{\partial x^k} + \tfrac{1}{2}\mathrm{W}_k^k = 0.$$

Now, for charge-free space $\mathrm{W}_k^k = 0$, so that we have at once

$$\frac{\partial s^k}{\partial x^k} = 0,$$

and further, as a consequence of differentiating the first, since we have

$$\frac{\partial^2 h^{rs}}{\partial x^r\, \partial x^s} = 0,$$

$$\frac{\partial \mathrm{W}^i}{\partial x^i} = 0.$$

The last equation is therefore satisfied, provided $W^1 = f(r)$ only, for we shall take

$$W^1 = W^2 = W^3 = 0.$$

If the variation vanishes on the boundary considered—*i. e.*, if we can neglect the boundary integral over the surface of the charges, since the variation is necessarily zero over the infinite boundary,—we have

$$\int \delta H \, dx = \int (W^i \delta \psi_i + \tfrac{1}{2} W^{ik} \delta g_{ik}) \, dx = 0;$$

so that

$$W^i = 0 \quad \text{and} \quad W^{ik} = 0,$$

and we get at once, exactly Maxwell's equation for free space:

$$s^i + \frac{\partial h^{ri}}{\partial x^r} = 0, \quad s^i = 0.$$

We have thus exactly verified Maxwell's equation for the conditions under which they are valid, since in that case we can neglect the surface integral of the flux over free space and near objects of low electrification.

Potentials and Intensities.

A fundamental calibration invariant is expressed by the following differential equation:

$$\left\{ \psi_i + \frac{\partial}{\partial x^i} \log g_{jk} \right\} = f_i \binom{q}{q'},$$

where $f_i \binom{q}{q'}$ is an arbitrary function.

The solution for the last differential equation for the ψ's allows us to solve for them when the q's are given, thus determining the character of our manifold completely. The fact that the q's allow us to determine the ψ's is at once evident from the fact that we have chosen the q's so that the paths of the particles are geodesics. Knowing the path of an electron around a positive nucleus would enable us to determine the law governing the forces acting on the electron; in fact, for the astronomical case it is precisely the above process of reasoning that led Newton to postulate the laws of gravitational attraction between material bodies.

Thus

$$\frac{\partial g_{jk}}{\partial x^i} + g_{jk} \cdot \psi_i = f_i \cdot g_{jk} = \bar{f}_{ijk},$$

which is our fundamental differential invariant.

For the differential fundamental form,

$$ds^2 = H_1^2\,dx_1^2 + H_2^2\,dx_2^2 \ldots H_4^2\,dx_4^2,$$

our equation becomes

$$\frac{\partial H_i^2}{\partial x^i} + H_i^2 \cdot \psi_i = f_i \quad \text{(not summed)}.$$

Applying integrability conditions, we obtain

$$\frac{\partial H_i^2 \psi_i}{\partial x^k} - \frac{\partial H_k^2 \psi_k}{\partial x^i} = \frac{\partial f_i}{\partial x^k} - \frac{\partial f_k}{\partial x^i} = F_{ik}.$$

From these equations results

$$\frac{\partial F_{ik}}{\partial x^j} + \frac{\partial F_{kj}}{\partial x^i} + \frac{\partial F_{ji}}{\partial x^k} = 0,$$

which is Maxwell's first set of equations, where the F_{ik} are the components of the intensity of the field; so that a knowledge of the function f_i would enable us to determine the electric and magnetic intensities of the field. We have, moreover, been able to derive all of Maxwell's equations from the consideration of invariant forms, which, with the previous discussion of the equations of dynamics, completes the determination of the phenomena of nature represented by our manifold.

From a consideration of the equations of dynamics, we can derive the form of the kinetic energy (so-called energy tensor) for an electron under the influence of a positive nucleus. Thus we find

$$-ds^2 = \left(1 - \frac{2e}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - \left(1 - \frac{2e}{r}\right) dt^2,$$

where e is a constant depending on the character of the nucleus: the only condition imposed on the nucleus is that it is very small in comparison with the path of the electron.

Thus

$$g_{11} = -\left(1 - \frac{2e}{r}\right)^{-1}, \quad g_{22} = -r^2,$$

$$g_{33} = -r^2 \sin^2\theta, \qquad g_{44} = \left(1 - \frac{2e}{r}\right).$$

We shall now proceed to calculate the ψ's, and thus derive the laws of force acting on the particle, through Maxwell's equation.

Taking our differential invariant in the form

$$g_{44} \cdot \psi_4 = f_4(r) = -\mathrm{V},$$

where V = potential of the nucleus, since the g's do not explicitly contain the time, and the propagated potentials $\psi_1 = \psi_2 = \psi_3 = 0$ for our particular case.

There is another important consideration as to why the potential should be of the above form. This choice for the potential is of weight *one*, while the ψ's themselves, assumed by Weyl as the components of the potential function, are of weight zero. The weight of an element is defined as the power of λ which the element is multiplied by if the calibration is changed—*e. g.*,

$$g_{ij} \longrightarrow \lambda g_{ij} \qquad \text{of weight one,}$$

$$g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \longrightarrow g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} \qquad \text{of weight zero.}$$

We shall now, since the choice of the potential is intimately connected with this argument, briefly review the reason* for taking the equations

$$\mathrm{G}_{ij} - \tfrac{1}{2} g_{ij}\mathrm{G} = -8\pi \cdot \mathrm{T}_{ij}$$

as the law of gravitation: or, since the argument for the choice of the potential is independent of the particular form of the equation, any set of equations of the form

$$\mathrm{G}_{ij} - \tfrac{1}{2} g_{ij}(\mathrm{G} - \lambda) = -8\pi \cdot \mathrm{T}_{ij},$$

where G_{ij} is the contracted curvature tensor and

$$\mathrm{G} = \mathrm{G}_{ij} g^{ij}.$$

The transition from discrete particles to the continuous medium involves, in Newtonian theory, the displacement of

$$\nabla^2\phi = 0 \quad \text{by} \quad \nabla^2\phi = -4\pi\rho_0.$$

We shall now find the corresponding modification of Einstein's equation

$$\mathrm{G}_{ij} = 0.$$

Writing the Schwarzschild line element in the form

$$ds^2 = -\left(1 + \frac{2m}{r}\right)(dx^2 + dy^2 + dz^2) + \left(1 - \frac{2m}{r}\right)dt^2,$$

where the origin is now arbitrary and r denotes the distance from the attracting particle to the element ds. The effect

* Eddington's Report, p. 59.

of a number of particles being additive, to a first approximation, we have

$$ds^2 = -(1+2\Omega)(dx^2+dy^2+dz^2)+(1-2\Omega)dt^2,$$

where Ω = Newtonian potential.

Consider a point 0 in the medium where the density is ρ_0, and with 0 as centre describe an infinitely small sphere. If we neglect the matter in the sphere, the equation $G_{ij}=0$ will be satisfied at 0. Hence, in calculating the value of G_{ij} at 0, we need only consider the values of G_{ij} due to the matter inside an infinitely small sphere of uniform density.

Since $\dfrac{\partial\Omega}{\partial x}$, etc., vanish at 0, we have

$$G_{ii} = -\tfrac{1}{2}g^{11}\frac{\partial^2 g_{11}}{\partial x_1^2}+\tfrac{1}{2}g^{22}\frac{\partial^2 g_{11}}{\partial x_2^2}+\tfrac{1}{2}g^{33}\frac{\partial^2 g_{11}}{\partial x_3^2} + \frac{\partial^2}{\partial x_1^2}\log\sqrt{-g};$$

and since

$$\frac{\partial^2 g_{11}}{\partial x_1^2}=\frac{\partial^2 g_{44}}{\partial x_1^2}=-\frac{\partial^2}{\partial x_1^2}\log\sqrt{-g}=-2\frac{\partial^2\Omega}{\partial x^2}$$

we have

$$G_{ii}=\frac{\partial^2\Omega}{\partial x_1^2}+\frac{\partial^2\Omega}{\partial x_2^2}+\frac{\partial^2\Omega}{\partial x_3^2}=-4\pi\rho_0,$$

and similarly

$$G_{ii}=-\pi\rho_0, \qquad i=1,\ 2,\ 3,\ 4.$$

Moreover, since G_{ii} is of weight zero, ρ_0 is of weight *zero*.

Now $$G=g^{ij}G_{ij}=(-G_{11}-G_{22}-G_{33}+G_{44})I$$

if we take the Galilean value of g^{ij} at the point 0, and where I is a unit scalar of weight -1, since g^{ij} is the weight of -1.

Thus $$G=8\pi\rho_0 I=8\pi I\,.\,T$$

and $$-8\pi I T_{ij}=G_{ij}-\tfrac{1}{2}g_{ij}G.$$

This equation shows that T is of weight zero, T_{ij} of weight one, T_k^i of weight zero.

Now consider the system of equations proposed by Einstein:

$$-8\pi I T_{ij}=G_{ij}-\tfrac{1}{4}g_{ij}G_0,$$

$$-8\pi I S_{ij}=G_{ij}-\tfrac{1}{4}g_{ij}G,$$

and $$T_{ij}=S_{ij}+\frac{1}{4I}(G-G_0)g_{ij},$$

where $$S_i^k=F_{ir}F^{kr}-\tfrac{1}{4}F^{rs}F^{rs}\,.\,\delta_i^k.$$

In order to be consistent with the weighting system shown above, the potential must be of weight *one*, for S_{ij} is necessarily of weight one, since T_{ij} and S_{ij} must have the same weight in any system. The importance of this aspect of the subject of weights was emphasized by Pauli (*Encycl.* 1921), and an attempt was made by F. Juttner (*Math. Annalen*, 1922) to modify the Einstein equation,

$$-8\pi S_{ij}=G_{ij}-\tfrac{1}{4}g_{ij}G,$$

so that the various terms became of the same weight.

L. P. Eisenhart has shown (Proceedings of National Academy, April 1922) that the ψ's in the second fundamental form are not the potentials of the field, but that the true potentials must be of weight *one*, whereas the ψ's are of weight zero. The choice of the potential V we have defined above satisfies this restriction, and is in agreement, therefore, with the geometry of paths. (Eisenhart and Veblen *.)

Differentiating with respect to g_{44}, we have

$$g_{44}\frac{\partial\psi_4}{\partial g_{44}}+\psi_4=-\frac{\partial V}{\partial g_{44}}$$

or

$$-\frac{\partial\psi_4}{\partial g_{44}}=\frac{\psi_4+\dfrac{\partial V}{\partial g_{44}}}{g_{44}};$$

so that

$$W^4=\frac{-{g_{44}}^2}{g_{44}\dfrac{\partial V}{\partial g_{44}}-V},$$

and the equation satisfied by the potential is

$$-g^2{}_{44}=\frac{1}{c^2}\left(\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}\right)\left(g_{44}\frac{\partial V}{\partial g_{44}}-V\right).$$

At

$$n=2e,\quad g_{44}=0;$$

so that a solution of the equation at this point is

$$V=\frac{\epsilon}{r}.$$

By a change of variable $z=\dfrac{1}{r}$, our equation becomes

$$\frac{1}{c^2}\left(\frac{1}{z^4}-\frac{2e}{z^3}+\frac{4e^2}{z^2}\right)=\frac{d^2V}{dz^2}\left\{\left(\frac{1}{2e}-z\right)\frac{dV}{dz}-V\right\}.$$

* Eisenhart and Veblen, Proc. Nat. Acad. Feb. 1922, p. 19.

Assuming a solution of the form

$$V = \Sigma a_n z^{m+n},$$

the equation may be written

$$\frac{1}{c^2}\left(\frac{1}{z^4} - \frac{2e}{z^3} + \frac{4e^2}{z^2}\right)$$

$$= -(m+n)(m+n-1)(m+l-1)a_n a_e z^{2m+n+l-2}$$

$$+(m+l)(m+n)(m+n-1)\frac{a_n a_e}{2e} z^{2m+n+l-3}.$$

i. Now, calculating the coefficient of $z^{2m+n+l-2}$, we have as the coefficient for $m=0$,

$$-n(l-1)(n-1)a_n a_e + \frac{l_n}{2e}(n+1)a_{n+1}a_e$$

$$+\frac{n(l+1)(n-1)}{2e} a_n a_{e+1}.$$

A solution appropriate to this expansion is

$$-V = \frac{1}{c^2}\left(\frac{r}{2} - 1\frac{e^2}{r} + \frac{8e^3}{r^2} + \ldots\ldots\right);$$

this expansion converges for large values of r.

Now we know that

$$\frac{4e^2}{c^2} = \epsilon,$$

ϵ = charge on the nucleus which determines the constant e.

The potential is equal to

$$V = \frac{\epsilon}{r} \text{ app.}, \quad r > 10^{-5} \text{ cm}.$$

When, however, the radius becomes of the order 10^{-5}, the first term begins to predominate. The significance of this term will be discussed further when we have developed the meaning of the equation.

Collecting coefficients in a different manner, we have

ii. $$-n(n-1)(l-2)a_n a_{e-1} - (n-1)(n-2)(l-1)a_{n-1}a_e$$

$$+nl(n-1)\frac{a_n a_e}{2e}$$

as the coefficient of z^{n+l-3}.

This gives as a solution for the potential

$$V = \frac{4e^2}{\epsilon c^2}\frac{1}{r^2} + \epsilon r - \frac{\epsilon c^2}{10e} r^2 \ldots\ldots,$$

where ϵ = charge on the nucleus.

This expansion, which converges only for very small values of r, gives the value of the potential inside the singular shell, which corresponds to the value of r at which the potential becomes infinite. We note that the intensity, which is negative near the interior side of this singularity, positive on the exterior side, approaches $\pm\infty$ at the boundary of this region.

There is a particular case, in which our equations for W^4 break down, which we shall now consider.

Suppose we have

$$\bar{f}_1=\beta\eta_{14}, \quad \beta=\text{const.},$$

then we have $$\psi_4=\beta,$$

and further $$W^4=\infty.$$

Now in this case we can calculate the forces at once from our invariant differential form, we have

$$\bar{f}_4=\beta\eta_{41}=-V,$$

and the components of force are

$$F_{4r}=\frac{2Be}{r^2},$$

the other being equal to zero.

The fact that $W^1=\infty$ does not enter into this discussion since the derivative $\dfrac{\partial H}{\partial\psi_2}$ does not exist in this case.

In this particular solution, for which Coulomb's law is verified, the calibration, in our sense, is uniform at every point.

F. Juttner in a recent article discusses this very case, and arrives at the same solution for a spherical body

$$-V=\beta\eta_{44}.$$

We may notice that the potential is zero at the boundary of the nuclear universe.

The writer (Proc. Nat. Acad. 1923) has shown that if the laws of mechanics hold and the space is Riemannian, *i. e.* the functions can be expressed in the form $\psi_i=\dfrac{\partial\rho}{\partial x^i}$ where ρ is a scalar function, then the law of conservation of energy holds provided

$$S_{ij}=T_{ij}+\Delta_{ij},$$

where Δ_{ij} has indeterminate principal directions and $\Delta^k_{ik}=0$.

Now the solution

$$V=\beta g_{44} \text{ or } \psi_4=\beta, \quad \psi_1=\psi_2=\psi_3=0,$$

satisfies this relationship, while the other solution which leads to the existence of the doublet structure cannot possibly satisfy this equation.

Thus, the inverse square law only holds in systems from which there is no radiation of energy—*i. e.*, Maxwell's equations are only valid for quasi-stationary motion which emits no radiation. When radiation takes place, these equations have to be modified.

These conclusions are in agreement with Bohr's dynamical laws which govern the motion of electrons. In open orbits, such as those described by an α-particle approaching the nucleus of an atomic system, the experiments of Rutherford show that the inverse square cannot hold.

From a consideration of the form of the differential equation, we see that the position of the pole is given by the equation

$$V=\frac{\partial \log g_{44}}{\partial r}\frac{\partial V}{\partial r}.$$

In order to prove the existence of this pole, take an infinitesimal region surrounding $r=0$, such that within that neighbourhood the following equality is satisfied:

(i.) $\quad V > -n\frac{\partial V}{\partial r}$, and hence $V < \frac{a}{r}$;

$$\therefore \quad \frac{2}{r}\frac{dV}{dr}+\frac{d^2V}{dr^2} > 0,$$

or

$$-\frac{g^2_{44}}{r\frac{dV}{dr}+V} > 0 \text{ within the neighbourhood of } r=0,$$

or

$$\frac{4e^2}{\frac{r^2}{a}} > 0, \quad a = \text{finite quantity},$$

$$r$$

which is impossible.

(ii.) $\quad V = -r\frac{dV}{dr}$,

$$g^2_{44} = 0,$$

which is also impossible; so that we must have

(iii.) $$V < -r\frac{dV}{dr}:$$

(*a*) $$\frac{dV}{dr} = -\sigma^2,$$

then the denominator of the right-hand side is positive, and hence a pole exists in the region $2e > r > 0$;

(*b*) $$\frac{dV}{dr^2} = +\sigma^2,$$

since at $r=2e$,

$$\frac{dV}{dr} = -\frac{\epsilon}{r^2},$$

$$\frac{dV}{dr} = \frac{d^2V}{dr^2} = 0 \text{ at some point in this region,}$$

which is impossible.

Therefore V becomes infinite within the region

$$2e > r > 0.$$

The experimental investigations of Rutherford on the bombardment of nuclei by α-particles show that the electric intensity becomes infinite in the region of the nucleus, $r < 10^{-12}$, which is in agreement with the character of the differential equation. The change in the sign of the intensity in the immediate vicinity of the singular shell accounts, moreover, for the fact that all the negative charges in the universe have not been neutralized.

Return, now, to a consideration of the solution in expansion within the singular shell; the potential is given by the series

$$V = \frac{4e^2}{\epsilon c^2}\cdot\frac{1}{r^2} + \epsilon r - \frac{\epsilon c^2}{10e}\cdot r^2 \ldots\ldots,$$

which approaches $\frac{4e^2}{\epsilon c^2}\cdot\frac{1}{r^2}$ as $r \doteq 0$.

Outside, we obtain the solution

$$-V = \frac{1}{c^2}\left(\frac{r}{2} - \frac{4e^2}{r} + \frac{8e^2}{r^2} - \frac{8e^4}{3r^3}\ldots\ldots\right),$$

which gives within the range $10^5 > r > 10^{-10}$,

$$V = \frac{\epsilon}{r} \text{ to a first approximation.}$$

When, however, $r > 10^5$,

$$V = -\frac{r}{2c^2}.$$

This anomaly is due to the presence of world electricity

in an analogous manner to the existence of world matter postulated for the maintenance of the universe.

When $r=2e=10^5$ app., $g_{44}=0$, *i. e.* the time component stands still while $g_{11}=\infty$, so that we have reached the end of the nuclear universe. The finite universe of the nucleus, viewed from an exterior point, has many properties common to the stellar universe of Einstein. Viewed from within, the universe seems to be of infinite spatial dimensions, while the time component seems slower and slower as we approach that infinite region becoming zero at infinite. The region beyond $r=2e$ is altogether shut off from us by a barrier of space and time.

If we take an electron on the boundary of this nuclear region $r=2e$, and let it move under the action of the nucleus, it will finally fall towards the nucleus, taking up a position on the singular shell beyond which it is unable to pass. We have, therefore, in our system a stable electric doublet composed of the nucleus and the electron, whose moment M

$$M=10^{-22} \text{ electrostatic unit.}$$

This stable doublet system is assumed by Sir J. J. Thomson in his work on electronic conduction. The effect of the external field on the direction of the doublets gave, as we have seen in the introduction, qualitative results which agree with experiment at all temperatures. A similar construction has been assumed by Abraham to account for high-frequency currents and radiation from antennæ. For high-frequency currents the electron is not shot out of the atom, but oscillates about the position of equilibrium, generating the electric waves which appear in this case. This model, as worked out by G. W. Pierce, has given results for the radiation from bent antennæ which agree well with experiment.

Conclusion.

From considerations based on Weyl's principle of invariance of calibrations, the writer has deduced the laws of force operative in the neighbourhood of the nuclear charge, which satisfy the results obtained by Rutherford on the deflexion of α-particles by atomic nuclei; the existence of the doublet used by Sir J. J. Thomson in his theory of metallic conduction; and, further, the important consequence that the inverse square law only holds for charges which do not radiate energy, which agrees with the dynamics of Bohr's atomic model.

In conclusion, the writer wishes to express his gratitude to Professor E. P. Adams for his interest and assistance during the writing of this paper.

CXIII. *On the Use of Medians for reducing Observations relating to several Quantities.* *By* Professor F. Y. Edgeworth, *F.B.A.* *.

HEREIN are submitted some further considerations on a method which the writer proposed in the 'Philosophical Magazine,' August 1887 and March 1888, for determining several quantities connected with observations by equations many times more numerous than the sought quantities. The facility and the accuracy of the method are now to be reconsidered.

I. The directions given by Professor Turner in the improved statement of the method which he contributed to the 'Philosophical Magazine,' December 1887, are very simple; yet in applying them to concrete examples he experienced difficulties. I attempt to lighten those difficulties by restating the *rationale* of the method, and by amplifying the directions for its application.

The use of the plural median is an extension of Laplace's method of *Situation* ('Théorie Analytique des Probabilités,' Supplement 2). The simplest example of that method occurs when there are several observations each purporting to afford an equally good value of the *quæsitum*—say, $v_1, v_2, \ldots v_n$, each equalled to hx. The rule is to arrange the values of x given by these equations in the order of magnitude—say, as points on a line; and to put for x a value which has as many of the given points above as below it. When the number of observations, n, is even, this condition is fulfilled by *any* point between the values of x numbered $\frac{n}{2}$ and $\frac{n}{2}+1$ in the order of magnitude. Now, let the coefficient h be different for different observations, while the error to which an observation is liable—the actual or absolute, as distinguished from the percentage, error—is the same for all the observations. We shall then have n equations of the form $h_1x-v_1=0, h_2x-v_2=0 \ldots$; from which n values of x may be extracted. These values—say, $x_1, x_2, \ldots x_n$—are then to be arranged in the order of magnitude, and each multiplied by the corresponding value of h. Then for x there is to be taken a value such that the sum of the expressions of the type $h_r(x-x_t)$, $x>x_t$ should as nearly as possible be equal to the sum of the quantities $h_s(x_s-x)$, $x<x_s$. In other words, the sum of the weighted

* Communicated by the Author.

residuals (using weight in a somewhat special sense), each taken in absolute quantity, say

$$\Sigma h_r \mid x - x_r \mid ,$$

is to be as small as possible. This condition is analogous to that which constitutes the Method of Least Squares, viz. that the sum of the squares of the said residuals should be a minimum (see as to the analogy, Phil. Mag. 1888, vol. xxv. p. 185). The required point may or may not coincide with one of the given points $x_1, x_2, \ldots x_n$. The coincidence is necessary when the sum of the products of type $h_r x_r$ is odd; and it becomes very probable, even when that sum is even, if the weights $h_1, h_2, \ldots$ are of very various sizes.

When we go on to the case of two *quæsita*, two dimensions in space, we have for the analogue of given points given lines of the type

$$x \cos \theta_r + y \sin \theta_r - v_r ;$$

each of these being multiplied by a coefficient corresponding to the worth of the observation, the sum of the weighted residuals, each in absolute magnitude, say

$$\Sigma h_r \mid x \cos \theta_r + y \sin \theta_r - v_r \mid ,$$

is to be as small as possible. If the data are in the form $a_r x + b_r y = c_r$, where $c_r = \sqrt{a_r^2 + b_r^2} \times v_r$, the v's being observations of equal worth (in the sense above defined): the expression to be minimised, viz.

$$\Sigma \mid a_r x + b_r y - c_r \mid ,$$

becomes identical with the above by putting $h_r = \sqrt{a_r^2 + b_r^2}$.

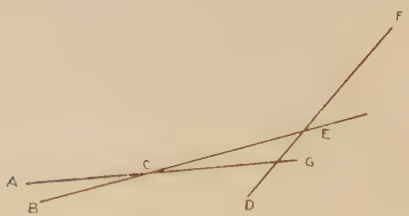
The search for the required point (x, y) may be facilitated by the following proposition. If through (x, y) is drawn a line parallel to the axis of x, and on it are marked the points $x_1, x_2, \ldots x_n$ at which it intersects lines given by the observations, then the point (x, y) is the median of the points $x_1, x_2, \ldots x_n$; and the like is true of a line drawn through (x, y) parallel to the axis of y. For the sum of the weighted residuals, which is to be minimised, may be written

$$\Sigma \mid x h_r \cos \theta_r + y h_r \sin \theta_r - h_r p_r \mid .$$

If now the median of the points of intersection is not x, let it be x'. Then, substituting x' for x in the above written expression, we shall obtain a smaller value for the sum. The proposition is not convertible. A point which fulfils the above condition secures what is in a certain sense a *minimum*

value, but not necessarily the *least possible* value for the sum of weighted residuals. Accordingly, the required point is not always secured by the rule at first proposed (Phil. Mag. July 1887), even as improved by Professor Turner (Dec. 1887). The intersection of the locus, consisting of points each of which is a median with respect to points on a horizontal with the corresponding locus for verticals, determines a minimum, but not always the *least possible* value for the sum of weighted residuals. Among the minima so determined the one which makes the sum of weighted residuals the *least possible* is to be selected. There is thus obtained what Professor Turner desiderated, an "additional criterion for extracting a single solution from the median loci" (*loc. cit.* p. 470). The trouble of selection is not likely to be considerable, since the compared points will probably be in the same neighbourhood, and accordingly the great part of the sum that is to be minimised will be the same for both.

Fig. 1.



For example, let the segment of the line AC from the point A where it intersects another line, not shown, up to C be a locus of double medians: meaning that if at any point on the locus a line is drawn parallel to x that point will be the median of the points given by intersection of the said line with the observation-lines, will be an x-median, say; and will also be a y-median. Let the line CE be likewise a locus of double medians. And let there be no other double-median points (or loci). The required point then must be on either AC or CE; and it will be on both, at their intersection C. For if it were at any other point on either of the segments, the sum of weighted residuals would be greater by the (weighted) perpendicular let fall from that point on the other line. Now let EF also be a locus of double medians (up to a point where it intersects with another line, not shown). By parity the required point must be either C or E. It will be the intersection of

those two lines for which the weighted perpendicular let fall from the intersection on the third line is the less. Thus if the weights are not very unequal, it will be E, and not C.

Even after the criterion has been applied, there may remain a certain indeterminateness when the condition of least possible sum of residuals is equally fulfilled by all the points on the segment of a line, or even all within an area. In the former case, on the analogy of the practice commonly employed for the simple Median, we should take for the sought point the bisection of the segment which is a double median. In the case of the locus being an area, on the same principle we should take the centre of gravity of the area.

The former case is seemingly not infrequent. For it may be expected that when a line that is the locus of x-medians crosses a line that is the locus of y-medians a segment of the x-median locus will sometimes be also a y-median. For commonly, where the weights are not very unequal, it is presumable that the number of observation-lines to the right of an x-median will be nearly the same as that on the left; and likewise the number of lines above a y-median about the same as the number of those below. Accordingly, it may be expected that, when an x-median line is intersected by any other line, the x-median locus after the intersection will be a segment of that second line: the segment which has the first line on the right [or left] after the intersection if before the intersection the second line was on the right [or left] of the first. *Mutatis mutandis*, a like proposition is true of a y-median. Whence it appears that intersecting median lines will sometimes be so related as to have a segment in common. For example, in fig. 1 if AC is a locus of x-medians and EF of y-medians it may be expected that CE will be a locus of both x- and y-medians. But if AC [or BC] were an x-locus and BC [or AC] a y-locus, then beyond the intersection the loci will continue on different lines, one on CG and the other on CE. Again, if AC is an x-median and GC a y-median, CE is probably the continuation of the x-median, BC of the y-median; and likewise if x and y are interchanged in this statement. If AC is an x-median and CE a y-median, or conversely, then it is likely that *both* segments are double medians. In this and the two preceding cases the solution is limited to a point.

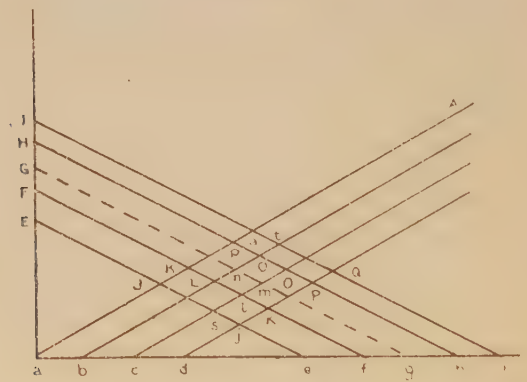
The double-median locus occurs in what is probably the simplest case of the problems under consideration: when the

observation-lines consist of two sets of parallel lines of the respective forms

$$x\cos\alpha+y\sin\alpha-p_r=0\ ;\ x\cos\alpha-y\sin\alpha-p_s=0$$

all of equal worth—there must be at least two sets for the reason given by Laplace: "si la marche de deux elements [two *quæsita* such as our x and y] était rigoureusement la même : ces éléments ne formeraient qu'une seule inconnue" (*Probabilités*, Introduction, ed. 3, § LXIV.). In fig. 2

Fig. 2.



let the number of observations at first be odd ; the dotted line Gg being included. Then the x-median locus will consist of the following segments:

$$ej,\ jk,\ kl,\ lm,\ mn,\ no,\ op,\ pq,\ qI$$

(ej and qI indefinitely produced at one extremity). Likewise the y-median locus is

$$EJ,\ JK,\ KL,\ nm,\ mO,\ OP,\ PQ,\ Qi$$

(JE and Qi produced indefinitely). The two loci have a common segment mn ; of which the middle point is to be taken as the solution of the problem.

If, the dotted line Gg being removed, the number of observations becomes even, each median locus will consist no longer of lines, but of spaces. The x-median locus will be (1) the space enclosed by the lines jd and je produced indefinitely, (2) the quadrilateral $jkls$, (3) $lLoO$, (4) the quadrilateral $opqt$, (5) the space between qI and qA. Likewise the y-locus will consist of spaces ; and the two loci will have in common the quadrilateral $lLoO$ (the dotted line being

absent) of which the central point is to be taken as the solution. This kind of indeterminateness is probably rare; at least as rare as the analogous incident in the case of the single median, a tract of line left vacant (the data being of various weights).

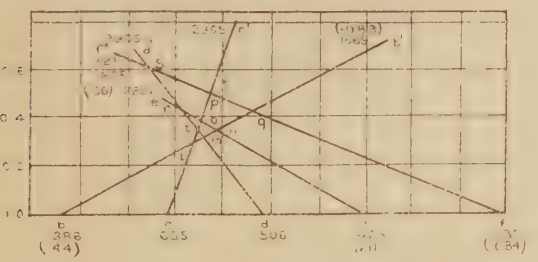
I subjoin some examples illustrative of the preceding statements.

Ex. 1. The first example does not require recourse to first principles. It consists of five equations in which the two sought quantities x and y are constants of a curve fitted by Professor Bowley to given statistics ('Journal of Royal Statistical Society,' 1902, pp. 340, 341). Observing several percentiles he connects with intervals, each equal to unity, between successive percentiles the linear function of x and y which would be equal to those intervals, theoretically if the series of statistics were indefinitely prolonged. There are thus obtained from the *apparent* percentiles the following five equations :—

$$(b)\ x\,0{\cdot}886 - y\,0{\cdot}1569 = 1,$$
$$(c)\ x\,0{\cdot}635 - y\,0{\cdot}2305 = 1,$$
$$(d)\ x\,0{\cdot}506 + y\,0{\cdot}3935 = 1,$$
$$(e)\ x\,0{\cdot}423 + y\,0{\cdot}722 \ = 1,$$
$$(f)\ x\,0{\cdot}335 + y\,0{\cdot}825 \ = 1.$$

(In the original there is another equation labelled (a); but as it is of very inferior worth, it is omitted here.) The others are not all of the same worth; but they may be treated as such for a first approximation.

Fig. 3.



The lines representing the equations are drawn in fig. 3 copied from Professor Bowley's fig. (*loc. cit.*). Underneath each line is placed (the numeral not in brackets) the coefficient of x in the line; and above each line the coefficient of y. Starting from the axis of x, let us first find the median

of the five points of intersection on that line. Whereas the sum of all the x-coefficients is 2·785, the half-sum 1·3925; since 0·886 is less than the half-sum, and 0·886 + 0·635 greater, the point c is the median. The like is true of every point on the line (c) up to l. At l the locus of x-medians changes to lm: since the x-coefficient of the only line to the left of lm is 0·635, less than the half-sum, and this added to 0·886 the coefficient for lm is greater than the half-sum. By parity, the locus does not turn at m, but goes on to n. The series of segments proves to be

$$cl,\ lm,\ mn,\ no,\ op,\ pc' \text{ (produced)}.$$

Likewise for the y-medians, whereas the half-sum of the coefficients is 1·169, for a vertical drawn through f the median is at f since 0·1669 + 0·2305 is less than the half-sum and the same *plus* the y-coefficient of the line (f) 0·825 is greater than the half-sum. The like is true of every point on that line up to q. Then the locus changes to qn. The series of segments is

$$fq,\ qu,\ no,\ or,\ rs,\ su \text{ (produced)},$$

u being the intersection of the line (e) and (f), not shown in the fig. The segment no being common to the two loci contains the required point; for which the middle point of the segment may be taken.

Next, let us take account of the fact that the observations are of unequal worth. Roughly (c) and (d) may be regarded as of equal worth, (b) and (e) as of only half, and (f) as of only a quarter of that worth. The bracketed numerals in the fig. show the altered coefficients. The half-sum of the x-coefficients is now 0·939; and accordingly c is still the median for the axis of x. The new locus is

$$cl,\ lm,\ mt,\ to,\ op\ pc' \text{ (produced)}.$$

Likewise the new y-locus is

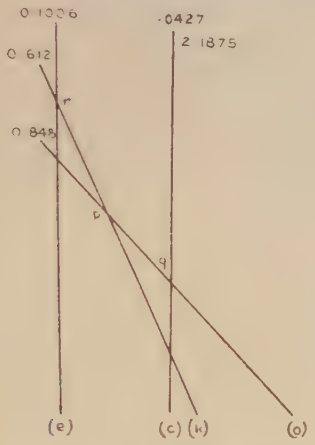
$$en,\ no,\ ot,\ tr,\ re' \text{ (produced)}.$$

The segment ot common to the two loci affords the solution.

Ex. 2. The next example consists of thirteen equations given by Laplace for the determination of x (Laplace's "z") and y, two constants pertaining to the elliptical figure of the earth (*Mécanique Celeste*, liv. iii., § 41; two of the fifteen equations there given being here provisionally omitted, as explained presently). The central portion of the network is shown in fig. 1. Whereas the x-coefficient for all the lines is identical, viz. unity, the x-median is the locus of points each of which has six lines on the right and six on the left. The lines being labelled (a) (b) (c) . . . . (o) in the order of

the intercepts which they make on the abscissa, and the lines (*i*) and (*j*) being provisionally omitted, the line (*g*) forms the *x*-median locus in the neighbourhood of the abscissa. Thence, continually turning corners, the locus has come to be, in the neighbourhood represented by the fig., qp, a segment of the line designated (*o*); followed by pr a segment of the line (*k*). The designations of the lines are given below, and their *y*-coefficients above. The half-sum of all the *y*-coefficients being 2·693, the sum of the coefficients for lines above pq excluding the coefficient of (*e*)

Fig. 4.



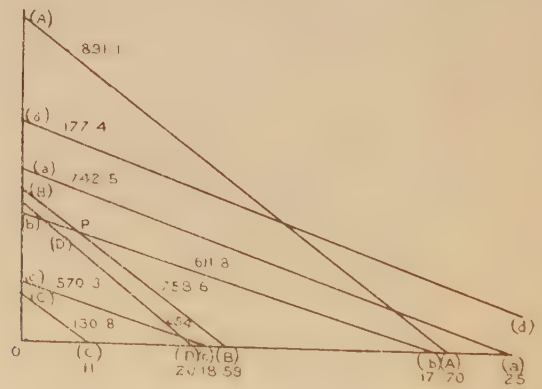
is 2·1875 as indicated on the fig. Accordingly, pq is a *y*-median, since $2{\cdot}230 = 2{\cdot}1875 + 0{\cdot}0427 < 2{\cdot}693$, $2{\cdot}230 + 0{\cdot}848 > 2{\cdot}693$. For a like reason pr is a *y*-median. The two loci do not go together beyond the points q and r. Accordingly, p, the meeting-point of the segments pq and pr, is the required determination. Its numerical value will be compared with Laplace's result presently.

Ex. 3. The next example is formed by eight equations cited by Thackeray in the Astronomical Monthly Notices, vol. xlvi. p. 335: for two corrections required in the calculation of Venus' semidiameter. The equations are, in the order and with a division given by Thackeray:—

(*a*) $25x + 742{\cdot}5y = 46{\cdot}27$	(A) $70x + 891{\cdot}1y = 112{\cdot}79$,
(*b*) $19x + 611{\cdot}8y = 29{\cdot}07$	(B) $59x + 758{\cdot}6y = 43{\cdot}91$,
(*c*) $18x + 570{\cdot}3y = 13{\cdot}23$	(C) $11x + 130{\cdot}8y = 2{\cdot}71$,
(*d*) $5x + 177{\cdot}4y = 14{\cdot}21$	(D) $20x + 254{\cdot}3y = 13{\cdot}87$.

As each of these equations is presumably formed by putting together in different numbers equations of equal worth, the sums as they stand may be taken as representing, by the coefficients, the relative worth of the data. Thus, the sum of the x-coefficients being 227, the x-median is the locus of the point which has a sum of x-coefficients as nearly as may be equal to 113·5 on the right and on the left. Likewise the locus of y-medians has as nearly as possible coefficients amounting to half 4136·8 above and below. Fig. 5 roughly represents the lines, the unit of y being ten times larger than that of x. The x-coefficients are written below the abscissa (with the exception of line (d)); and the y-coefficients above

Fig. 5.



each line. Adding up the x-coefficients from the left we have up to b exclusive the sum 108 and, b being included, 127. Thus bp is part of the x-median locus. At p, the point where it meets B, the locus changes to B: the sum on the left exclusive of B being now $68 < 113·5$: while $68 + 59 > 113·5$.

For the y-median, adding the coefficients from the top downwards along the axis of y, we have the sum up to (B) exclusive 1811 and, (B) being included, 2569·6. Thus the segment of line (B) above the point p is a y-median, as well as an x-median. At the intersection of (B) with (a) the loci part company. Going back to the segment pq of line (b), q being the intersection of line (b) with line (A) not shown in the fig., we find that it is a y-median (as well as an x-median). For the sum of the y-coefficients above (those of (a), (d) and (A), viz. 1811, is less than the half-sum 2068; while if the coefficient of B, viz. 611·1, is added, the sum becomes greater than 2068. At q both loci change to (A).

They part company at the next intersection (not shown in the fig.) between (A) and (B). We have then on the principle above explained to choose between the points p and q; the choice depending on the question which is the shorter of the two perpendiculars, that from p on (A) or that from q on (B), each multiplied by the corresponding weight. In other words, if $x_1\ y_1$ are the coordinates of p, and $x_2,\ y_2$ those of q, the question is which is the smaller, $70x_1+891{\cdot}1y_1-112{\cdot}79$ or $59x_2+758{\cdot}6y_2-43{\cdot}9$. Performing the calculation, I find that the answer is in favour of q.

Ex. 4. The next example consists of 29 equations relating to the same corrections as the x and y of example 3, kindly submitted to me by Professor Turner. The incident that the coefficient is the same, viz. unity, for all the equations (presumably of equal worth) facilitates the tracing of the x-median locus (as in Ex. 2). But the selection of the optimum point is embarrassed by the close relation between the two loci, which turn several corners together.

Ex. 5. A still more marked appearance of indeterminateness is presented by an example taken from Laplace's *Mécanique Céleste*, § 41: seven equations involving two constants entering into the elliptical form of the earth as determined by observations on the meridian at different places. When account is taken of the different weight (in the sense above defined, p. 1075), the equations may be written (as they are by Bowditch in a note to his translation)—x being here substituted for z in the original:—

$$\begin{aligned}
88448{\cdot}69 - \ & 3{\cdot}4633x & = 0,\\
34834{\cdot}78 - \ & 1{\cdot}3572x - 0{\cdot}40928y = 0,\\
42072{\cdot}94 - \ & 1{\cdot}6435x - 0{\cdot}65651y = 0,\\
61624{\cdot}50 - \ & 2{\cdot}4034x - 1{\cdot}11856y = 0,\\
275793{\cdot}15 - 10{\cdot}7487x - & 5{\cdot}59932y = 0,\\
84071{\cdot}71 - \ & 3{\cdot}2734x - 1{\cdot}79546y = 0,\\
27495{\cdot}85 - \ & 1{\cdot}0644x - 0{\cdot}89289y = 0.
\end{aligned}$$

Here the fifth line owing to its preponderance proves to be the median locus both for x and y *throughout*. The problem thus becomes: to find the point on that line for which the sum of the weighted perpendiculars on the other lines is the least. We have thus to determine x and y such that the sum of the left sides in (six of) the above-written equations, each taken in absolute quantity, should be the least possible. But since the point $(x,\ y)$ is on the fifth line we may substitute

the value of x in terms of y derived from the fifth equation for x in each of those six equations; and thus we are presented with a problem of the ordinary kind relating to a single median. Performing the requisite operation, I find for y 483.

If this is not in close agreement with Laplace's result, viz. 246·9, it should be remembered that the paucity of the observations and the preponderance of one of them renders the probable error incident to any calculation very considerable. It suffices to observe that the seven values of y which Laplace finds (by a process described below) range from -221 to $+1252$; and that the weight of one of them is nearly equal to the weight of all the others put together.

The working of the above examples was facilitated by some devices which may be of general application: (1) Whereas it is convenient to transform the coordinates (or one of them) to an origin in the neighbourhood of the sought point, this may often be effected with sufficient accuracy as follows. Distribute the equations into two classes, say, as in Example 3, if the number of the equations was larger, according to the greater or less slope of the lines. Take the medians of the x-coefficients, of the y-coefficients, and of the absolute term, for all the equations of the first batch, therewith to construct a derived equation in x and y. Likewise from the second batch construct a second derived equation. The two derived equations treated as simultaneous furnish approximate values of x and y. (2) Or we may take for the first approximation that point which has as many lines on the right as on the left, and as many above as below. Professor Bowley has with good effect thus treated our example 1 (*loc. cit.*). (3) When the network of lines is much entangled it is convenient to omit some of them provisionally, and to determine an approximate value with the remainder. I thus in Example 2 omitted equations (i) and (j) which by their proximity threatened entanglement. With the remaining thirteen I found for x 99680·5 and for y ·00551. Transforming to that point as origin, I found for the locus of double medians the segment of a line, the middle point of which is $x=99679·7$, $y=·00562$. Which is in satisfactory accord with Laplace's result, viz. $x=99676$, $y=·00567$, and the slightly different result which Bowditch, correcting Laplace, I think correctly, substitutes (*loc. cit.*).

The process applied to Example 5 is analogous as to its latter part with Laplace's treatment of the case. But he obtains a first equation whereby to eliminate x (his :), not as a median locus, but by the ordinary method of Least Squares.

In cases like our fourth and fifth this partial use of Least Squares, as well that method in its entirety, has an advantage over our method in respect of what Professor Turner calls "straightforwardness." This advantage is no doubt considerable in the case of several variables, loci to be traced in many-dimensioned space. But the choice of methods cannot be decided without consideration of their relative accuracy. In cases where the method of Medians was known to be much more accurate than that of Least Squares the former might be preferable, even though more laborious as well as less "straightforward."

II. The accuracy of the double Median depends on much the same considerations as those which relate to the single Median. Accordingly, the interest of this second section extends beyond the special object of the article. The comparison of the (single or double) Median method with that of Least Squares is prejudiced by two misapprehensions exaggerating (*a*) one the defects of the Median, (*b*) the other the merits of Least Squares

(*a*) It is presumed that determation by way of Medians is less exact because it sometimes leaves the segment of a line, or even (in the case of the double Median) a space within which no unique value is distinguished. But the comparative definiteness of the Arithmetic Mean is illusory, considering that the determination is liable to a probable error. It has been shown (Phil. Mag. 1888, p. 187) that in the case of the single unweighted Median the tract of line left vacant is likely to be a very small fraction—of the order $1/\sqrt{n}$, where n is the number of equally good observations—of the probable error incident to the Arithmetic Mean. The corresponding proposition for the weighted Median is *à fortiori*. It may likewise be shown that the probability of the vacant tract being as large as the said probable error is extremely small. These propositions may be transferred from the single to the double Median through the proposition that the point, or locus of points, for which the sum of the weighted residuals is the least possible must be an invariant, not varying with the choice of axis. Accordingly, if a large tract of line (or *mutatis mutandis* a space) were left indeterminate, by a proper choice of coordinates, with a point on the tract as origin, the case may be shown to imply the improbability that the Median of numerous (sporadically dispersed) points on a line should be inderminate over a large tract. The tract of indeterminateness left vacant by the Median may indeed *seem* to be greater than the probable error of the Arithmetic Mean. But that is probably because the probable error of

the Arithmetic Mean is greater than it is represented to be by the usual formula (the square root of the mean square of residuals divided by n or $n-1$, and multiplied by 0·67450). For that formula presumes that the observations are *independent*, uncorrelated, which in the concrete is probably often far from true—for instance, observations of the kind which form our Example 3 (as pointed out by Stone, Astronom. Monthly Notices, vol. xxv. p. 57). In short, the Median not only reminds us that our results are liable to a probable error: it may also warn us that the probable error is greater than would commonly be supposed—at least as great as the tract left indeterminate by the Median.

(*b*) The Method of Least Squares enjoys an undue preference in virtue of its connexion with the Normal Law of Error. For probably that law is not in general fulfilled by observations so perfectly as to justify the preference given. The preferability varies with the character of the observations. I proceed to prescribe one or other of the compared methods for different cases, without knowing in what proportion these cases occur in practice. Three types may be distinguished:

(1) The normal law of error may be fulfilled with the perfection proper to games of chance, and occasionally realized by concrete statistics, *e. g.* measurements of human stature (Yule, 'Theory of Statistics,' ch. xv. § 13). In this case the Arithmetic Mean prescribed by the Method of Least Squares is decidedly preferable on grounds of Inverse Probability. But it should be remembered that even in this case the certain advantage of the Arithmetic Mean is slight; the probable error of the Median is only about 25 per cent. larger than that of the Arithmetic Mean (cp. Yule, *op. cit.* ch. xvii. § 3).

(2) Probably a commoner case is where the normal law is fairly well fulfilled by the observations, about as well, say, as in the examples cited by Merriman ('Method of Least Squares,' § 31), by Airy ('Errors-of-Observation,' sub finem), or by Chauvenet 'Astronomy,' vol. ii., Appendix). In this case the advantage of the Arithmetic Mean over the Median is less. It is much less, if, as probable, the imperfection consists of a feature which is specially favourable to the Median, the abnormal protuberance of the extremities of the curve which represents the observations.

(3) When the curve representing the observations is quite abnormal it is very possible that the Median should have the advantage in respect of accuracy. Not that the Method of Least Squares is inapplicable in this case. For—*pace* some distinguished writers—the method, as prescribed by Laplace,

does not demand affinity to the normal form in the distribution of the original crude observations. The normal form which is characteristic of the method is presented not by the crude observations, but by (series of) their *means* (cp. article on "Probability," *op. cit.* § 134–5). But the like is, in general, true of Medians. This kind of *mean*, as well as the Arithmetic, is distributed according to an ancillary normal frequency-curve. Which method is preferable depends on the character of the respective ancillary curves: which of the two has the smaller modulus (or "standard deviation"). Laplace in his second Supplement has shown cases in which the method of "situation," the single Median, is preferable to the Method of Least Squares. He has recommended the method of situation not only by precept but by example; having employed it in the determination of constants pertaining to the elliptical form of the earth (*Mécanique Céleste*, liv. iii. § 40, 41). Bowditch, the able translator of the *Mécanique Céleste*, says in a note referring to the passage cited: "This method, proposed by Boscovich, and peculiarly well adapted to the present problem, is not now so much used as it ought to be" . . . . "We shall hereafter find, in several instances, that the method of Least Squares, when applied to a system of observations in which one of the extreme errors is very great does not generally give so correct a result as the method proposed by Boscovich." More generally, when the extremities of the curve representing the crude observations are abnormally protuberant, the Median is apt to be preferable. When very abnormal conditions are apprehended the Median may well be employed to check and control, if not to replace, the ordinary method.

In short, the use of the Median (single or double) is often easier, and sometimes more accurate, than the Method of Least Squares. Nevertheless, the feeling of physicists and mathematical statisticians in favour of the usual practice is no doubt founded not merely on familiarity, but on deeper unexpressed reasons. Some may be suggested. The probable error for the ancillary normal curve pertaining to the Median, depending on the height of the ordinate—the aggregation of the observations—at the centre, is not so readily ascertainable as the corresponding coefficient for the Method of Least Squares. In order that there should exist an ancillary normal curve, it must be postulated that there is an aggregation about the centre; whereas no such postulate is required for the rival method. The absence of the requisite central aggregation may be serious in a case with which Laplace has introduced the Method of Least Squares

(*op. cit.* liv. ii. ch. 4), and which he has employed with effect—in a manner probably admitting of wide application—to discover laws masked by fortuitous happenings, such as the average difference in the height of the barometer at different times of day (*op. cit.* liv. ii. ch. 5). The case is that in which the crude observations are represented by a rectangle; one value being as likely as another between finite limits—say, $\pm a$.

Altogether, we may conclude with Laplace that, in certain cases the Method of Situation is preferable to the Method of Least Squares, the Method called *par excellence* "the most advantageous method" (*op. cit.* Supplement 2, ed. 1847, p. 619); that (in general) it is natural to consider the Median as "a very good approximation" (*op. cit.*, Introduction, p. lxix). "Mais le résultat donné par la methode la plus avantageuse est encore préférable."

CXIV. *The Motions of Electrons in Gases and the Formation of Negative Ions in Air.* By Leonard B. Loeb *.

IN a recent note in this magazine Dr. V. A. Bailey †, in commenting on the writer's work ‡ reported in 1921, appears to have been under an erroneous impression as to the method of measurement employed by the writer. In the paper referred to the writer attempted to verify the J. J. Thomson theory of ion formation by studying the mobility curves in air obtained below 10 cm. pressure, using the Rutherford alternating-current method with alternations of *square wave form* obtained from a commutator. This method is the method of experimentation termed by Dr. Bailey the "Lattey modification" of the Rutherford method. Thus the writer did *not* use the method termed by Bailey the "Franck modification," which involves the use of an alternating current of *sinusoidal* form, as stated in Bailey's note.

In the work quoted above the equations used for a verification of the theory by the writer were but a rough approximation to the rigorous equations, as owing to the uncertain knowledge of many of the factors a greater precision did not seem warranted. The agreement obtained was merely a qualitative one. Lately much more careful measurements have been made, using the *square wave oscillations*, with the added advantage that some rough measurements on the mobilities of electrons in air at the pressures

* Communicated by the Author.

† Bailey, V. A., Phil. Mag. xlvi. p. 213 (July 1923).

‡ Loeb, L. B., Phys Rev. n. s. vol. xvii. p. 89 (Feb. 1921).

employed in this work were also made. The results of these investigations, which are shortly to appear in print, enabled the use to be made of the rigorous theory of J. J. Thomson. The results obtained by including the values of the electron mobilities, as measured directly, in the rigorous attachment equations appear to give a striking verification of the theory. When, however, certain corrections which seem theoretically necessary are made on the values of the electron mobilities the agreement initially observed is completely vitiated. In the application of the corrections to the electron mobilities, certain assumptions are made about the space charge effects due to the accumulation of ions which may not be fulfilled in air. Thus it is still doubtful whether the corrected electron mobilities are more justified than the uncorrected ones. It is therefore, at present, an open question whether the *rigorous form* of the Thomson theory is correct or not.

The writer agrees with Dr. Bailey that it is possible that "h," the attachment constant of J. J. Thomson, may decrease as the velocity of thermal agitation "u" decreases, though he is unaware of any direct evidence that this is so. He desires to point out that for the change in "u" produced by *higher* fields H. B. Wahlin * found no evidence of a marked change in "h." It is also desired to point out that in order of magnitude "h" is such a distinct function of the chemical nature of the gas that even were the assumption of Bailey correct (viz., that "h" is a function of "u"), the theory of Thomson would be applicable in its broad outlines.

The writer has unfortunately long been aware of the fact that the use of sinusoidal alternating potentials of high frequency for the measurement of electron mobilities is an unsatisfactory mode of procedure. Yet up to the present the method has furnished the only means of estimating the electron mobilities in most gases above 100 mm. pressure. It might be pointed out that where the results obtained by the writer using sinusoidal oscillations overlap the very beautiful results of Townsend, Bailey, and their collaborators, they are roughly in agreement with them, as will be seen in a forthcoming paper by K. T. Compton †.

Department of Physics,
University of California,
Berkeley, Calif.
Aug. 25, 1923.

* Wahlin, H. B., Phys. Rev. n. s. vol. xix. p. 173 (Feb. 1922).

† Compton, K. T., "Mobilities of Electrons in Gases," Phys. Rev. n. s. vol. xxii. (1923) (to appear).

CXV. *On the Motion of Electrons in Gases.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

I HAVE received a letter from Dr. L. B. Loeb with which he encloses a copy of a note he has sent to the Philosophical Magazine, taking exception to a statement I made on his work in connexion with the formation of negative ions and the measurements of the velocities of electrons in gases.

My statement consisted of seven lines, and as Dr. Loeb has written several papers on these subjects, no reader would suppose that I had related all Dr. Loeb's work. The main facts to which I alluded, however, are those published by him in the Philosophical Magazine of January 1922, and in my paper I gave that reference.

In his paper of January 1922 Dr. Loeb gives reasons which led him to suppose that previous determinations of "mobilities" were unreliable, and states that in his latest experiments on the determination of the mobility he used "*high-frequency oscillations from an audion oscillator.*" He thus obtained a value of the "mobility" which was 2800 cm./sec. for nitrogen, which he points out is much higher than the value 200 cm./sec. given by other experimenters, and he adopts the higher number as the fundamental basis of the calculations of the chance of ion formation which he gives in this paper.

Under these conditions I am surprised to find the following sentence in Dr. Loeb's note to the Philosophical Magazine: "Thus the writer did *not* use the method termed by Bailey the 'Franck modification,' which involves the use of an alternating current of *sinusoidal* form, as stated in Bailey's note," as the high-frequency oscillations from an audion oscillator are usually supposed to be of sinusoidal form.

I have therefore looked up another paper by Dr. Loeb (Physical Review, p. 24, 1922) where he gives a diagram and full details of his experiments with the audion oscillator, and it is clear from this paper that the apparatus is designed to generate oscillations of sinusoidal form, and they are treated as such in the investigation.

Dr. Loeb appears not to have noticed the fact that I referred to his paper in the Philosophical Magazine of January 1922, where he gives the impression that "mobilities" determined by sinusoidal alternating forces are reliable.

The statement in Dr. Loeb's note that the method of determining electron "mobilities" by means of sinusoidal alternating potentials has up to the present "furnished the only means of estimating electron mobilities in most gases above 100 mm. pressure," is by no means correct. Townsend's method, involving the magnetic deflexion of a stream of electrons moving in a uniform electric field, may be used to determine the velocity of electrons in a gas at any pressure when no ions are present. Most of the determinations, it is true, have been made with gases at pressures below 100 mm., because the smaller the pressure the smaller is the magnetic force required to deflect the stream, and it is easier to work with small magnetic forces than with large forces involving the use of coils with iron cores. From the measurements that have been made it has been found that the velocity in the direction of an electric force Z is a function of the ratio Z/p, so that if the velocities are determined for one value of Z and one value of p, they may be considered as being known for any multiples nZ and np of this force and pressure.

Yours truly,
V. A. Bailey.

Electrical Laboratory,
Parks Road, Oxford.
Sept. 13, 1923.

CXVI. *On Atomic Structure and the Reflexion of X-Rays by Crystals. By* D. R. Hartree, *B.A., St. John's College, Cambridge* *.

§ 1. *Introduction.*

In a recent paper † I have given an account of an attempt to determine approximately the field inside and in the neighbourhood of an atom by quantitative analysis of the terms appearing in the optical and X-ray spectra, following the theory given in a more qualitative form by Bohr. It was pointed out there that the dimensions of the orbits and variation of time along them could be calculated approximately from the results of this analysis and applied to the problem of the intensity of the reflexion of X-rays by crystals, and some preliminary results of this application to sodium were quoted and compared with Bragg's experimental results ‡.

* Communicated by R. H. Fowler, M.A.

† D. R. Hartree. Proc. Camb. Phil. Soc. vol. xxi. part 6 (1923).

‡ W. L. Bragg, R. W. James, and C. H. Bosanquet, Phil. Mag. xli. p. 309; xlii. p. 1 (1921); xliv. p. 433 (1922). These papers will be referred to as B. J. B. i, ii, and iii.

The object of the present paper is to consider this application in more detail, and also to inquire whether and to what extent it may be possible to obtain from X-ray reflexion evidence on the orientations of the orbits in the atom and on the relative phases of the electrons in the different orbits

For simplicity we will consider a rocksalt crystal, as it is for rocksalt that the experimental work has been done, and it is quite sufficient for the purpose of this paper. With the arrangement of atoms in the rocksalt crystal, the different possible reflecting faces can be divided into two groups: namely, those for which every plane of atoms parallel to the reflecting face is similarly constituted, *i.e.*, contains equal numbers of Na and Cl atoms regularly arranged, and those for which alternate planes are differently constituted, *i.e.*, contain Na atoms only and Cl atoms only. The former are faces for which one index is even, the latter those for which all indexes are odd; they will be referred to as "even" and "odd" faces respectively.

The formula for the intensity of X-ray reflexion need not be quoted here in its entirety. The important part of it for the purposes of this paper is the factor, written F^2 in Bragg's papers, which depends on the number and arrangement of electrons in each scattering atom. If the wavelets scattered by the individual electrons are all in phase, then F is equal to the number of electrons concerned in the scattering; this is the case when the electrons are distributed in a space small compared with the wave-length of the incident X-rays, and also in certain other conditions. When the wave-length of the incident X-rays is of the same order as the distances between the various electrons, the scattered wavelets are not in general all in phase, and the amplitude of the wave scattered by the atom as a whole will be less than that scattered when the wavelets are in phase; this is expressed by a smaller value of F. The extent of this interference depends on the electron arrangement and on the angle of scattering.

The scattering by the individual electrons is assumed to be properly deduced from classical electrodynamics: the results so obtained are at any rate of the right order, as pointed out in Bragg's paper*. The amplitude of the wavelet scattered by one electron according to classical electrodynamics will be taken as the unit of amplitude; the value of this unit in terms of the mass and charge of the electron etc., and also the summation of the contributions

* B. J. B. iii. p. 446.

from the whole of the crystal, is taken into account in the coefficient of F^2: we are here concerned only with the pure number F which is the property of a single atom or a mean property of the atoms.

We will consider the incident wave to be plane, and the interference to be taking place at infinity. The wave reflected from the whole crystal is the resultant of the waves reflected from each plane of atoms in the crystal parallel to the reflecting face: if the reflexion conditions are satisfied then all the waves reflected from similar planes are in phase and those reflected from different planes have definite phase relations. Moreover, each wave reflected from a plane is made up of the waves scattered by each atom.

If any one electron is situated at a distance x from the plane parallel to the reflecting plane, containing the centre of the atom to which it belongs, then the wavelet scattered by it for glancing angle β differs in phase by $(4\pi x \sin\beta)/\lambda$ from that scattered by an electron in the plane $x=0$, which is conveniently taken as zero of phase. The summation of the effects of the wavelets scattered from different electrons is best done by means of the complex variable; since the amplitude of the wavelet due to one electron is taken as unity and the phase difference from standard is $(4\pi x \sin\beta)/\lambda$, then if

$$z=\Sigma \exp\left[(4\pi i x \sin\beta)/\lambda\right], \quad . \quad . \quad . \quad (1{\cdot}1)$$

$|z|$ is the amplitude and $\arg z$ is the phase of the resultant vibration, the summation being taken over all relevant electrons. Similarly for a distributed charge, if de is the charge lying between planes x and $x+dx$,

$$z=\int \exp\left[(4\pi i x \sin\beta)/\lambda\right] de. \quad . \quad . \quad . \quad . \quad (1{\cdot}2)$$

We will first calculate F for a model atom in which the actual moving distribution of electrons has been averaged out into a spherically symmetrical distribution of charge (§ 2), and then for an atom in which the plane of the orbit of each electron is supposed to be fixed, but the electron distribution is averaged out in the plane of each orbit (§ 3). We will then show under what conditions this averaging process is justified (§ 4), and evaluate F for an atom model which is less simple than either of those considered in § 2 and § 3, but is probably a nearer approximation to reality (§ 5). A short review of the experimental results (§ 6) is followed by a consideration of the numerical results obtained from the use of the formulæ established in §§ 2–5 (§ 7).

§ 2. *Spherically Symmetrical Atoms.*

We will consider first an atom in which each orbit has been averaged out into a spherically symmetrical distribution of charge.

A circular orbit radius r will average out into a uniformly charged spherical shell of the same radius with total charge 1, and the contribution to F is $(\sin \xi)/\xi$ per electron*, where

$$\xi = (4\pi r \sin \beta)/\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

λ being the wave-length of the X-rays and β the glancing angle.

If the electron in a non-circular orbit spends a fraction df of the radial period between r and $r+dr$, then this orbit averages out into a spherically symmetrical volume distribution in which the charge enclosed between radius r and $r+dr$ is df. The contribution of this shell to F is $(\sin \xi/\xi)df$ where ξ is given by the above formula, and as the waves scattered by all the shells are obviously in phase, the total contribution of a non-circular orbit is $\int (\sin \xi/\xi)df$, so altogether

$$F = \Sigma \oint \frac{\sin \xi}{\xi} df. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}2)$$

the summation being taken over all orbits †.

If, as for the even faces of rocksalt, each reflecting plane contains equal numbers of the two kinds of atoms, then, since the waves from each kind of atom must be in phase, the values of F can be added directly; and if the molecule and not the atom is taken as the unit of structure the value of F to use in the reflexion formula is the sum of the values of F for the separate atoms. Obviously for spherically symmetrical atoms F is a function of the angle of reflexion only and does not depend on the face from which reflexion takes place.

§ 3. *Planes of Orbits Fixed.*

We will next consider a less drastic process of averaging, and treat the planes of orbits as fixed. The circular orbit then averages out into a ring distribution of charge. If ϕ is the angle between the normal to the plane of the orbit and the normal to the reflecting face, and r, θ are polar coordinates on the plane of the orbit, then $x = r\cos\theta\sin\phi$;

* See B. J. B. iii, p. 439.

† For a circular orbit r and hence ξ is constant over the whole orbit, and $\oint df = 1$.

also in formula (1·2), $de=d\theta/2\pi$, for the ring is uniformly charged and carries a total charge of unity.

Hence putting as before

$$(4\pi r \sin \beta)/\lambda=\xi,$$

we have

$$z=\int_0^{2\pi} \exp\,[(4\pi ir \cos\theta \sin\phi \sin\beta)/\lambda]d\theta/2\pi$$

$$=\int_0^{2\pi} \exp\,(i\xi \cos\theta \sin\phi)d\theta/2\pi=J_0(\xi \sin\phi). \quad (3\cdot1)$$

Since the non-circular orbits are multiply and not singly periodic, they average out to a sheet of charge symmetrical about the nucleus, and it is easy to see that the contribution given by such a sheet is

$$z=\oint J_0(\xi \sin\phi)df, \quad . \quad . \quad . \quad . \quad . \quad (3\cdot2)$$

where as before df is the fraction of a period spent by an electron between r and $r+dr$.

Since the contributions from all electrons are in phase (for z is purely real for them all),

$$F=\Sigma\oint J_0(\xi \sin\phi)df, \quad . \quad . \quad . \quad . \quad . \quad (3\cdot3)$$

and, as for the spherically symmetrical atom, the values of F are to be added numerically if each reflecting plane is similarly constituted and contains more than one kind of atom.

§ 4. *Justification of Averaging Process.*

We must now consider what may happen in actual fact: obviously each atom is not actually spherically symmetrical at each instant, and even if the mean distribution of the electrons in the atom over the lapse of a long time does give a spherically symmetrical distribution of charge, it is not justifiable to average out the electron distribution first and then calculate the reflected intensity; the instantaneous reflected intensity should be calculated and then averaged with respect to time.

It is fairly obvious that if we are going to consider the instantaneous reflected intensity we can no longer assume that the atoms of the same kind all have the same spatial distribution of electrons. Even if the motions of the electrons in a given orbit * were the same and in phase for all atoms,

* *I. e.* in an orbit specified by given quantum numbers.

a given wave reaches different atoms at different times and so would be scattered differently by each atom: actually there seems no reason to imagine any necessary relation between the phases of the electrons in different atoms, though such relations almost certainly exist between the electrons in a single atom.

Now consider all the atoms of one kind in one single reflecting plane, and for each atom consider the electron in some given circular orbit. Since the only coordinate which occurs in the formulæ is the perpendicular distance from the electron to the plane of nuclei, we can consider all the atoms moved in the plane till their nuclei are coincident. Then, if the electron is equally likely to be in any direction from the nucleus, the result will be a uniform spherical shell of charge, since the number of atoms in each plane is very large. This shell can then be divided up again into a series of uniform shells each with a total charge unity, one to each atom. Hence the unjustifiable averaging with respect to time for a single atom has been replaced by a valid averaging at each instant of time over all atoms of the same kind in each reflecting plane.

Similar considerations apply to the non-circular orbit.

If the plane of the orbit is supposed fixed with respect to the crystal, then a similar averaging over all atoms of the same kind in one reflecting plane gives the uniform ring distribution of charge for the circular orbit, as considered in § 3.

In § 3 use had to be made of the property of the non-circular orbit that it was multiply periodic, in order that it should average out into a sheet with circular symmetry. It is now only necessary to suppose that the directions of the axes of the instantaneous ellipses of the motions of the electrons in the different atoms are distributed at random in the plane of the orbit at any instant. It is possible that this might not be the case if the orbits were simply periodic.

As a partial confirmation of the argument it is interesting to take the results obtained by assuming the plane of the orbits fixed, and then average over all possible orientations of the plane, each direction being equally likely, so that the points in which the normal to the plane cuts the unit sphere cover it equally densely all over; the results should of course be the same as for spherical distribution. It is sufficient to consider a circular orbit for which, when the plane of the orbit is fixed,

$$\sigma = J_0(\xi \sin \phi). \qquad (3{\cdot}1)$$

The area of the unit sphere between ϕ and $\phi+d\phi$ is $\sin\phi\, d\phi$, so the averaging over all possible directions of the normal gives

$$z=\int_0^{\pi/2} \mathrm{J}_0(\xi\sin\phi)\sin\phi\, d\phi \quad . \quad . \quad . \quad . \quad (4{\cdot}1)$$

Putting $m=0$, $n=-\frac{1}{2}$ in the general formula *

$$\mathrm{J}_{m+n+1}(x)=\frac{x^{n+1}}{2^n\Gamma(n+1)}\int_0^{\pi/2}\mathrm{J}_m(x\sin\phi)\sin^{m+1}\phi\cos^{2n+1}\phi\, d\phi,$$

and substituting in (4·1), we have

$$z=(\pi/2\xi)^{\frac{1}{2}}\mathrm{J}_{\frac{1}{2}}(\xi)=(\sin\xi)/\xi,$$

agreeing with the result of § 2.

So the calculation for the spherically symmetrical atom can be carried out in two steps, first averaging over all possible positions of the electron in a given plane of the orbit, and then over all possible planes. The importance of this division is that the second averaging can now be restricted and not taken uniformly over the whole sphere. This will be considered in the next section. Meanwhile it should be mentioned that only for the spherically symmetrical atom is F independent of the crystal face, and it is only in such a case that the method used by Bragg to separate the effects of the two kinds of atoms in rocksalt is valid.

§ 5. *Atom with Planes of Orbits restricted but not fixed.*

As has just been shown, if we imagine the direction of the normal to the plane of a given orbit in the various atoms of one kind to be entirely at random, then the crystal reflects as if each atom were spherically symmetrical, and the reflexion can be calculated for this case, and also for the case when the normal to a given orbit is supposed fixed in the crystal.

We must now consider a little more closely the structure of the atom and of the crystal, and see if there is any reason to expect the normal to the plane of a given orbit to be unrestricted or restricted, and in the latter case what sort of restriction to expect.

There are two separate points involved—the possible orientation of the orbits in a single atom with respect to some axis in that atom, and the possible orientation of this

* See G. N. Watson, 'A Treatise on the Theory of Bessel Functions,' (Camb. Univ. Press, 1922) p. 373.

axis with respect to the crystal axis. If either of these orientations is unrestricted the crystal will reflect accurately as if each atom were spherically symmetrical. It will probably be accepted that in each atom the orbits are definitely orientated with respect to some axis; the question is rather, are the directions of these atomic axes distributed at random or are they restricted?

If the axis of each atom is restricted in direction at all, the most reasonable thing is to expect it to be parallel to one of the crystal axes, but as there is no difference between the six faces of an ordinary cubic crystal, for example, the directions of the atomic axes must be distributed equally among the two senses of the three crystal axes.

Turning now to the orientations of the orbits with respect to the axis of the atom, Bohr * considers that in a group of four electrons of the same total and azimuthal quantum numbers (such as occurs in Na and Cl), the instantaneous planes of the four orbits are arranged parallel to the sides of a tetrahedron which is definitely not regular and only has one axis of symmetry, which is aligned along the axis of the atom. If this tetrahedron were regular there would be no unique axis about which rotation would be likely to occur rather than about any other, and it might be expected to set itself with its dyad axes (which are mutually perpendicular) parallel to the crystal axes. But as the tetrahedron is deformed so as to have only one axis of symmetry, it may be expected to be capable of rotation as a whole about this axis: or, what is more important from the present point of view, it may be considered that for different atoms the azimuth of the tetrahedron about the axis of symmetry is unrestricted. Then averaging over all atoms whose axes are parallel to a given crystal axis, a circular orbit averages out into a distribution of charge which is best described as that due to a uniform ring of charge rotated about the axis of the atom.

To calculate the scattering by this distribution of charge, let ϕ_0 be the (fixed) angle between the normal to the plane of a given circular orbit of radius r and the axis of an atom. Take the crystal axes as axes of x, y, z, and let the axis of the atom be parallel to the x axis. Then the ring of charge must be supposed to be equally likely to be in any azimuth about this axis. Let l, m, n be the direction cosines of the normal to the face of the crystal; then it is easy to see that for a given azimuth ψ of the ring, the angle ϕ_1 between the

* N. Bohr, 'The Theory of Spectra and Atomic Constitution' (Camb. Univ. Press, 1922) pp. 93, 95.

normal to the ring and the normal to the crystal face is given by

$$\cos\phi_1 = l\cos\phi_0 + m\sin\phi_0\sin\psi + n\sin\phi_0\cos\psi, \quad (5{\cdot}1)$$

while by §3 the contribution of this ring at this azimuth to the scattering is

$$z = \mathrm{J}_0(\xi\sin\phi_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3{\cdot}1)$$

$$\xi = (4\pi r\sin\beta)/\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

As the ring is to be rotated uniformly about the axis of x, *i. e.* ψ is equally likely to have all values between 0 and 2π, we obtain as the mean contribution of the given orbit from the atoms which have their axes parallel to the x axis,

$$z = \int_0^{2\pi} \mathrm{J}_0(\xi\sin\phi_1)\,d\psi/2\pi.$$

Since the atoms are considered to have their axes distributed equally along the three crystal axes, the mean contribution of a given orbit to F, over all the atoms of the same kind, is

$$z = \int_0^{2\pi} \tfrac{1}{3}[\mathrm{J}_0(\xi\sin\phi_1) + \mathrm{J}_0(\xi\sin\phi_2) + \mathrm{J}_0(\xi\sin\phi_3)]\,d\psi/2\pi$$

$$= \mathrm{I}_{\alpha\beta\gamma}(\xi), \quad . \quad . \quad . \quad . \quad (5{\cdot}2)$$

where $\cos\phi_2$, $\cos\phi_3$ are obtained from (5·1) by cyclical interchange of l, m, n. Obviously I will depend on l, m, n, *i. e.* on the face of the crystal from which reflexion is taking place; it is convenient to take the indices α, β, γ of the face as suffixes to I, as they are integers while l, m, n are generally irrational. We have of course

$$l, m, n = \frac{\alpha,\beta,\gamma}{(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (5{\cdot}3)$$

For the whole atom the value of F becomes

$$\mathrm{F}_{\alpha\beta\gamma} = \Sigma\, c\int \mathrm{I}_{\alpha\beta\gamma}(\xi)\,df, \quad . \quad . \quad . \quad . \quad (5{\cdot}4)$$

where as before df is the fraction of a period which the electron spends between radii r and $r+dr$, and $\xi = (4\pi r\sin\beta)/\lambda$.

It is more convenient to deal with the spherical model as standard and work in terms of deviations from it; for this purpose we define a new function

$$\Delta_{\alpha\beta\gamma}(\xi) = \mathrm{I}_{\alpha\beta\gamma}(\xi) - \frac{\sin\xi}{\xi}, \quad . \quad . \quad . \quad . \quad (5{\cdot}5)$$

so that

$$\mathrm{F}_{\alpha\beta\gamma} - \mathrm{F}_0 = \Sigma\, c\int \Delta_{\alpha\beta\gamma}(\xi)\,df, \quad . \quad . \quad . \quad . \quad (5{\cdot}6)$$

where F_0 is the value of F calculated for a spherical atom.

On expanding the Bessel functions in (5·2) and performing the integration with respect to ψ, it appears that the term in ξ^2 in I is independent of ϕ_0 and of l, m, n, and moreover it disappears entirely when $(\sin \xi)/\xi$ is subtracted from I, so that Δ is fourth order in ξ at the origin. This is one reason why it is so convenient to work with.

Expanding further we find

$$\Delta_{\alpha\beta\gamma}(\xi) = \frac{\xi^4}{192} g(\phi_0) \left[\frac{\alpha^4+\beta^4+\gamma^4}{(\alpha^2+\beta^2+\gamma^2)^2} - \frac{3}{5} \right] + O(\xi^6) \quad . \quad (5{\cdot}7)$$

where

$$g(\phi_0) = 1 - 5\sin^2\phi_0 + \frac{35}{8}\sin^4\phi_0.$$

This shows the order of magnitude of the difference between the values of F for different faces for small glancing angles. It cannot be used for quantitative comparison with experiment, as higher order terms become appreciable, in the case of the outer orbits of the chlorine atom, for values of ξ not much greater than the minimum available value * for NaCl. Higher order terms rapidly become very complicated, and actually it is easier to evaluate I and Δ by numerical integration than from the series. However, from this formula we can obtain some rough information about the difference between the different faces of the crystal. The maximum and minimum values of $l^4+m^4+n^4$, subject to the restriction $l^2+m^2+n^2=1$, are given by $l^2=1$, $m=0$, $n=0$ (or a cyclical interchange of the three letters), and $l=m=n$. These correspond to the crystal faces (100) and (111) respectively, so the intensities reflected from these two faces, which are the most important ones in practice, will be the furthest removed from that due to a symmetrical atom, and in opposite directions: whether even so the difference is appreciable can only be settled by numerical evaluation of Δ.

The separation of the fourth order term in Δ into factors, one depending on ϕ_0 only and the other on the indices of the face only, appeared at first sight as pure accident, but it can be shown to be necessary. Obviously the integral of Δ over values of l, m, n corresponding to a uniform covering of the unit sphere must vanish identically; but for the fourth order term l, m, n can only appear in the combination $l^4+m^4+n^4$, whose average value over the sphere is 3/5. A similar simplification does not occur in higher order terms, as l, m, n can occur in more than one combination.

* See § 8.

§ 6. *Experimental Results.*

The experimental results suitable for comparison with the formulæ deduced above are those of Bragg, James, and Bosanquet * on the reflexion of X-rays by rocksalt. There are really two points to be investigated: the relative intensities of the beam reflected from different faces, and the absolute intensities. The first point is only considered rather briefly in the papers referred to. But if the intensity of reflexion actually depends on the face from which reflexion takes place, as well as on the glancing angle, the effect is worth investigation for two reasons at least:—

1. It may give information about the orientation of orbits in the atom or of atoms in the crystal;
2. It vitiates the method used to separate the effects of the two kinds of atoms;

so it seemed worth while to look into the matter more closely.

For this purpose Bragg's original uncorrected results were taken, and $A \sin \beta$ plotted against $\sin \beta$, where A^2 is the observed intensity relative to the (100) reflexion, and β is the glancing angle as before. The object of plotting $A \sin \beta$ instead of A is that it is more nearly constant, and it seemed probable that this plot would be one of the most sensitive to the effect that was being looked for.

The results are shown in fig. 1; the figures by each plotted point indicate the order of the reflexion and the face from which it takes place, following Bragg's convention of multiplying the indices of the crystal face by m for an mth order reflexion, *e.g.* (333) indicates a third order reflexion from the (111) face.

For high intensity reflexions the mechanical nature † of the crystal face has a very large effect on the intensity of reflexion, so not much weight must be given to the apparent differences between the faces shown by the (100) (110) and (222) reflexions; but for larger glancing angles the effect is small, and here there is at least a suggestion of a difference between different faces: taking the (100) face as standard, the points for the (330) and (444) reflexions are high by an amount of the order of 15 per cent. or of 30 per cent. in observed intensity. On the other hand, the point for the (220) reflexion agrees with the curve for the (100) face. Also for the odd faces the point for the (311) reflexion appears to be low. On the whole it seems that the results may be significant as regards differences between faces, but cannot be considered certainly so.

* B. J. B. i and ii.

† This term is meant to denote the state of the crystal face in regard to such properties as degree of polish, warping of the crystal planes, etc.

Figs. 1 and 2.—A sin β plotted against sin β, where
β =glancing angle,
A^2=reflected intensity relative to (100) face.

Fig. 1.

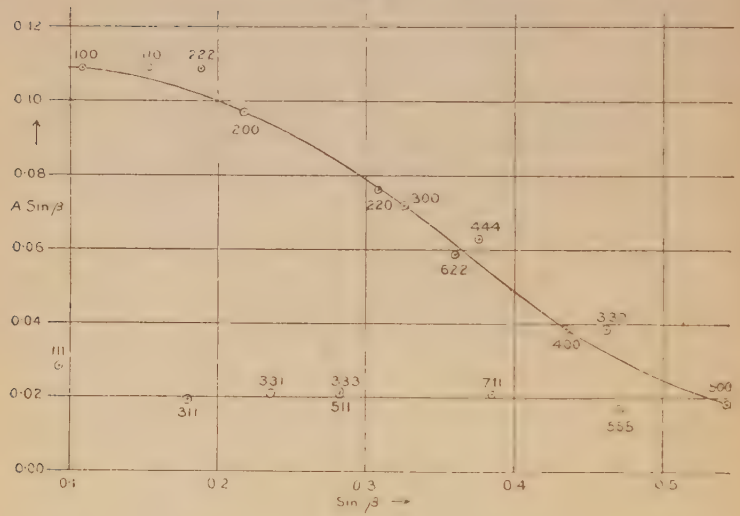


Fig. 2.

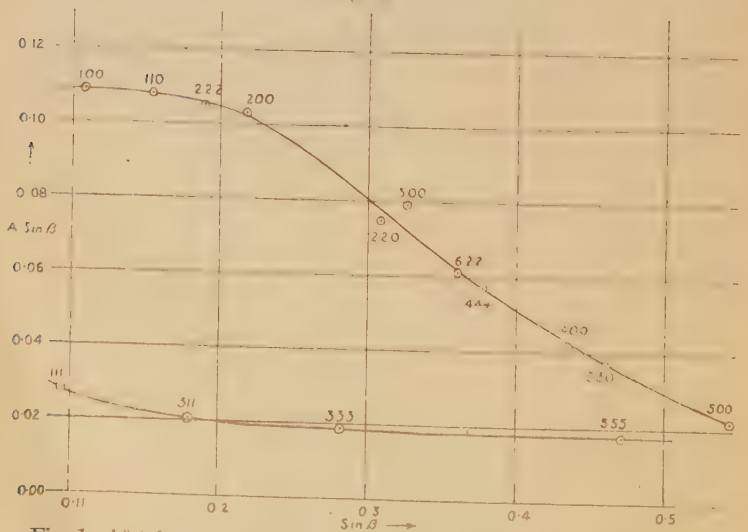


Fig. 1. A^2 taken direct from Bragg's original results.

The curve for even faces is drawn through the points for the (100) face. No curve is drawn for the odd faces as the number of points for any one face is not sufficient to define a significant curve.

Fig. 2. A^2 corrected for 'face effect' (§ 7.2).

Points for different faces have been given equal weight in drawing the curves.

It appears probable that the Debye correcting factor is a function of the glancing angle only, and does not depend on the face, or only slightly, and the same may be true of the extinction correction, so it does not seem that these two effects would alter the conclusion suggested by the last paragraph, that the intensity of reflexion may actually depend on the face from which reflexion takes place as well as on the glancing angle.

The absolute values of F given by Bragg are affected by uncertainty about the mechanical nature of the crystal face, and also about the real form of the correction for heat motion, more seriously than the relative values for different faces: the former uncertainty is serious at small glancing angles and the latter at large. There is now also the further doubt of the method used to separate the effects of the two atoms.

§ 7. *Numerical Results.*

The dimensions of the orbits and variation of time along them, which are necessary in order to make a quantitative comparison of theoretical with experimental results, have been obtained by analysis of the terms of the optical and X-ray spectra in the way already mentioned. For Na the atomic field so determined has been used directly; for Cl the field was extrapolated from those of K and Ca, and then adjusted to the X-ray spectrum of Cl; the calculated orbits must therefore be considered as less reliable than those for Na; the doubt is principally concerned with the outer part of the atom.

In figs. 3 and 4 $\frac{df}{dr}$ is plotted as a function of r for the non-circular orbits of the ions of Na and Cl as calculated in this way. The radii of the circular orbits are also shown. This function is all that is necessary for the calculation of the X-ray reflexion; the actual orbit is not required.

§ 7. 1. *Atomic Radii.*

Before going on to the intensity of X-ray reflexion by crystals, it is interesting to compare the dimensions of the atoms as calculated by the above methods with the values of the "radius of the atom" deduced by Bragg from the packing of the atoms in crystals *. For electropositive elements these values are only upper limits for the radius of the outer group of electrons present (on Langmuir's theory), which corresponds roughly to the radius of the smallest

* W. L. Bragg, Phil. Mag. xl. p. 169 (1920).

Figs. 3 and 4.—Calculated curves of $\frac{df}{dr}$ for X-ray orbits of Na and Cl.

df = fraction of a complete period spent by an electron between r and $r+dr$.

Unit of r is the radius of the 1_1 orbit of the hydrogen atom (0·532 A.U.).

Calculated radii of circular orbits are shown by short vertical lines cutting the axis of r.

Fig. 3.

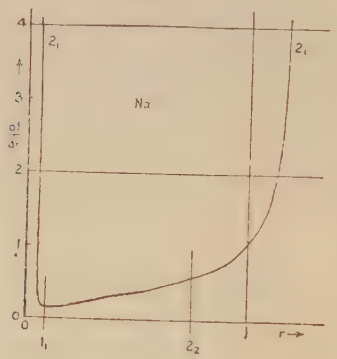
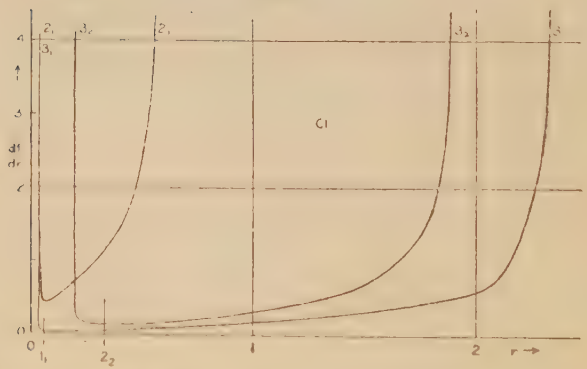


Fig. 4.



sphere enclosing all orbits of the ion (on Bohr's theory). For the electronegative elements Bragg's values are probably slightly low estimates for this radius, and for the inert gases * approximately correct estimates.

For Na, Bragg's value is 1·8 A.U., for Ne 0·65 A.U. The maximum calculated radius for the Na ion is that of the 2_1 orbit, and is 0·65 A.U., this is smaller than Bragg's upper

* Obtained by extrapolation from the preceding electronegative elements.

limit and just the same as his value for Ne ; the actual radius of the ion would be expected to be slightly smaller for Na than for Ne, but the agreement is surprisingly good considering the entirely different means of obtaining the estimates. The maximum calculated radius of the 3_1 orbit, the normal orbit of the valency electron, is 2·6 A.U. ; this is greater than Bragg's upper limit and suggests very strongly (apart from other evidence) that the atom is ionized, for the normal 3_1 orbit would penetrate well into the Cl atom in rocksalt. If the Na atom is not ionized, the 3_1 orbit must be profoundly modified, and it is assumed in all calculations to follow that the atoms are present as ions.

For Cl, Bragg's value from the packing in crystals is 1·06 A.U., the maximum calculated radius of an orbit normally present in an ion is 1·22 A.U. ; considering the extrapolation required to obtain the latter value, its agreement with Bragg's cannot be considered bad, especially as his value is probably rather an underestimate.

§ 7. 2. *Differences between Faces.*

In order to obtain numerical estimates of the differences between different faces it is necessary to specify the value of ϕ_0, the angle between the normal to the plane of an orbit and the atom axis, for each orbit. As already mentioned, Bohr considers that the four orbits of the same quantum number are arranged parallel to the sides of a tetrahedron with one axis of symmetry which is aligned parallel to the axis of the atom. Since the amount and even the direction of the distortion from a regular tetrahedron is unknown, the tetrahedron will be taken as regular with a dyad axis parallel to the crystal axis, the object of the distortion being simply to define an unique axis about which the tetrahedron is considered to be capable of rotation. Then $\sin^2\phi_0 = 2/3$ for all orbits occurring in groups of four with the same quantum numbers. On account of the high order of zero of the Δ functions at the origin, the K orbits contribute nothing appreciable to the difference from a spherically symmetrical atom, and so the value of $\sin\phi_0$ for them need not be specified.

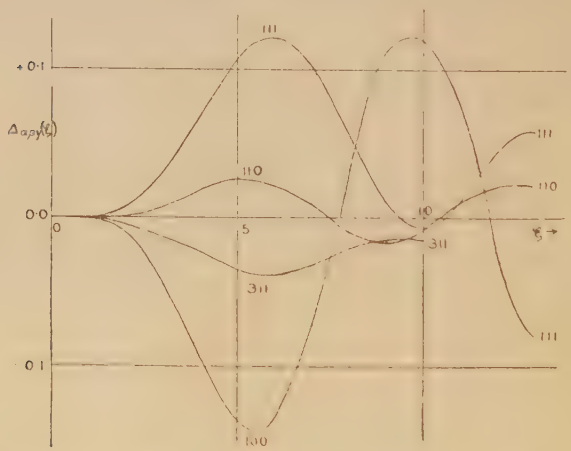
The values of the Δ functions for the (100) (110) (111) and (311) faces are shown in fig. 5. It will be seen that the difference between any pair of these four faces remains constant in sign till ξ is greater than 7, which is just about the maximum value it takes in the case of the sodium atom, with values of the glancing angles used in the experiments.

So that the contribution of the Na atoms has always the sign of the first term in the expansion for Δ.

From the dimensions of the orbits and the Δ functions, the differences of the values of F for different faces can be calculated; some figures are given in § 7.3. From these, rough corrections to the original observations can be obtained. To avoid as far as possible the difficulties connected with extinction and the Debye correction for heat motion, the correction was applied as follows. The factor necessary to reduce the theoretical results for a face to those for spherically symmetrical atoms was calculated for each face

Fig. 5.

Curves of Δ functions for faces (100), (110), (111), (311).



for a series of glancing angles and plotted, and each value of A [the square root of the observed intensity relative to the (100) face] multiplied by the value of this factor read for the appropriate glancing angle. Then, as before, $A \sin\beta$ was plotted against $\sin\beta$.

The results are shown in fig. 2, and on comparing with fig. 1 it certainly seems that the general agreement has been improved. But further experiments are necessary before it can be definitely said whether this is significant.

The results have been re-analysed into the effects due to the Na and Cl atoms separately, using the sum and difference curves determined by the observed values corrected for "face effect" as in fig. 2, and the results differ from Bragg's by quantities up to 0·5 in F.

§ 7.3. *Results for Sodium.*

Table I. gives values of F calculated for several models of the sodium atom, for comparison with observed values. In all cases the dimensions of the orbit have been taken to be as calculated in the way already mentioned.

The atom models are :—

A. Spherically symmetrical atom,

(1) $2\ 1_1$; $4\ 2_1$; $4\ 2_2$ orbits.
(2) $1\ 1_1$; $4\ 2_1$; $5\ 2_2$ orbits.
(3) $2\ 1_1$; $2\ 2_1$; $6\ 2_2$ orbits.

B. Atom with normals to planes of orbits fixed parallel to crystal axes.

C. Atom as defined in § 7.2, *i. e.* normals to planes of orbits making fixed angles ϕ_0 with the axis of the atom but capable of rotation round that axis ; atom axes parallel to the three crystal axes and distributed equally among them. $\sin^2\phi_0 = 2/3$.

For B and C the electron distribution is $2\ 1_1$; $4\ 2_1$; $4\ 2_2$ orbits.

The entries for A are actual values of F, for the other models differences from A (1) are given. The "observed" values depend of course on the analysis of the experimental results into the effects of the two atoms separately. Values tabulated as "obs. 1" are taken from Bragg's paper, those given as "obs. 2" are obtained from the results corrected for the "face effect" as described in the previous section. They are not given for the smallest glancing angle, as here the exact method of correcting for extinction is important and the results might not be comparable with Bragg's. The observed values should be compared with the theoretical values in the row A (1), as there is little doubt that the electron distribution there assumed is correct.

TABLE I.

		Crystal face.	Values of F or $F_{\alpha\beta\gamma} - F_0$. $\sin\beta = 0{\cdot}1$	0·2	0·3	0·4	0·5
	Obs. (1).		8·32	5·40	3·37	2·02	0·76
	Obs. (2).			5·76	3·84	2·03	0·82
Atom Model.	A (1).		8·87	6·18	3·41	1·66	1·18
	A (2).		8·76	5·81	2·70	0·66	0·04
	A (3).		9·01	6·58	3·79	1·70	0·78
	B [difference from A (1)]	100	−0·01	−0·15	−0·35	−0·83	−1·01
		111	0·01	0·21	0·45	1·02	1·61
	C [difference from A (1)]	100	−0·01	−0·08	−0·30	−0·60	−0·81
		311	0·00	−0·03	−0·09	−0·20	−0·27
		110	0·00	0·00	0·05	0·12	0·15
		111	0·00	0·05	0·21	0·45	0·68

It is very noticeable that the atom model with orbits in fixed planes gives larger differences between faces than atom C. This is to be expected since atom C gives an average distribution of charge more nearly uniform over a sphere than atom B, but the differences between atom C and the spherical atom are still very appreciable. The result A (1), (2), (3), shows that if the dimensions of the orbits are assumed, the results, for large glancing angles especially, are rather sensitive to the distribution of electrons among the orbits, but this distribution can probably be taken now as almost certain, and the experimental results looked on as possibly giving information about the dimensions of the orbits rather than about the electron distribution among them. Considering the many uncertainties, the agreement between theory and experiment for the orbital dimensions used does not seem at all bad.

§ 7.4. *Results for Chlorine.*

Table II. gives the results for chlorine in the same general form as Table I. gives those for sodium. For chlorine no results have been calculated for an atom model with planes of orbits fixed, as the results for sodium indicate that the difference between faces for such an atom would be too large, and also the model C is probably a better approximation. The values * of F for the spherically symmetrical atom model A are only given for the distribution of electrons :— $2\,1_1$; $4\,2_1$; $4\,2_2$; $4\,3_1$; $4\,3_2$.

TABLE II.

		Crystal face.	F or $F_{\alpha\beta\gamma}-F_0$. $\sin\beta=$ 0·1	0·2	0·3	0·4	0·5
	Obs. 1.		12·72	7·85	5·79	4·40	3·16
	Obs. 2.			8·23	6·05	4·25	3·30
Atom Model.	A.		13·96	8·99	7·88	7·03	5·37
	C−A.	100	− 0·08	0·62	−0·74	−0·06	0·00
		311	− 0·01	−0·16	−0·23	−0·20	
		110	0·00	0·12	0·11	0·01	0·05
		111	0·04	0·47	0·73	0·46	0·43

The difference between calculated and observed values is much greater than for sodium, the observed values being too small. The difference is much larger than the effect of

* The irregular differences between the values of F in successive columns of the table is due to the small number of values of $\sin\beta$ for which F is tabulated. The curve of F against $\sin\beta$ is quite smooth.

allowing for the difference between faces, it seems to be bigger at large glancing angles than at small, and is greater than two units in some cases.

The contributions of the different orbits to the calculated value of F are interesting. They are shown in Table III.

TABLE III.

		Contribution to F.				
Orbit.	$\sin\beta=0{\cdot}1$	0·2	0·3	0·4	0·5
2 1_1	2·00	2·00	1·99	1·97	1·94
4 2_1	3·83	3·36	2·67	1·86	1·13
4 2_2	3·91	3·68	3·32	2·84	2·27
4 3_1	1·76	−0·27	+0·20	+0·08	−0·03
4 3_2	2·46	0·22	−0·10	0·28	+0·06
Total	13·96	8·99	7·88	7·03	5·37

The small contributions from the 3_1 and 3_2 orbits at large angles are very noticeable. This is connected with the large range of r which they cover. The contribution of any one orbit is $\int\rho[(\sin\xi)/\xi]dr$ where $\xi=(4\pi r\sin\beta)/\lambda$; and if the orbit is sufficiently large and eccentric, then at large glancing angles the range of integration will include several maxima and minima of $(\sin\xi)/\xi$, and the greatest weight will be given to those of least numerical value (since df/dr increases nearly all the way from inside to outside of the atom), so the total contribution will be very small. As already mentioned, the dimensions of the outer parts of the orbits are the most uncertain points in the calculated dimensions of the Cl orbits, but fortunately this uncertainty affects F at large glancing angles only very little for the reason given, and the big difference in this region between theoretical and observed values cannot be ascribed to it.

Two possible explanations of the difference are suggested, viz.:—

(1) The frequencies of the electrons in 1_1 orbits in Cl are becoming appreciable compared with the frequency of the incident X-rays, and those electrons may be scattering less than a free electron on classical electrodynamics; if this were the case, the electrons in other $k=1$ orbits would perhaps, in the inner part of the orbit, also contribute less than the calculated amount to the scattering. The effect would be much less in sodium, for which the frequency of the electron in the 1_1 orbit is less, and also there are fewer other $k=1$ orbits. To avoid large powers of 10 the frequencies ρ divided by cR can be compared (c = velocity of light in vacuo,

R = Rydberg number) and the values are approximately as follows:

	ν/R or p/cR.
X rays used (Rh K_α).	1500
Cl 1_1 orbit	500
Na 1_1 orbit	200

so that the frequency in the 1_1 orbits of Cl is about a third of the frequency of the incident X-rays.

(2) It is very possible that the simple form used for the correction for the heat motion is not correct. It may well be, for example, that the value of F for the two atoms should be multiplied separately by different "Debye factors" before adding or subtracting to obtain the effective value of F for the odd or even faces.

§ 7.5. *Results for Potassium.*

Values of F at various glancing angles have also been calculated for potassium. The principal object of this calculation was to estimate the intensity of the (111) reflexion from sylvine. Since the K and Cl ions contain the same number of electrons in orbits of roughly the same dimensions, the odd order reflexions from the (111) face must be very weak, as is well known; but since the orbits have not exactly the same dimensions it would not be expected that the intensity of this reflexion would be absolutely zero, and it appeared worth while to calculate its value and see what information could be obtained from an experimental measurement of it.

For the first order reflexion from the (111) face of sylvine the calculated values of $F_K - F_{Cl}$ and $F_K + F_{Cl}$ are about 0·95 and 31 respectively; the principal part of $F_K - F_{Cl}$ is due to the difference between the outer parts of the 3_1 orbits in K and Cl, and it is rather sensitive to the difference of the maximum radii of these two orbits; as already mentioned (§ 7.4) the calculation of the orbits of the Cl ion is most uncertain just in this region, and the measurement of the (111) reflexion from sylvine would be very valuable in giving further information; the dimensions of the orbits of the K ion are more certain.

The ratio of the intensities of (111) and (100) reflexions with the above values of F is about 1 : 800; the ratio of the intensities of the (333) and (100) reflexions comes out to be 1 : 6000, and of the (555) and (100) about 1 : 10000.

A similar experiment on NaF would also be interesting.

§ 8. *Concluding Remarks.*

In all the formulæ for F given in this paper, the actual dimensions of the atom only appear in the quantity

$\xi = (4\pi r \sin\beta)/\lambda$, independent of the precise atom model used. Now for the reflexion at glancing angle β, λ and β are connected by the Bragg relation,

$$2\pi d \sin\beta = n\lambda,$$

where d is the distance between successive reflecting planes in the crystal, so that F must be thought of not as a function of λ and β separately, but of $(\sin\beta)/\lambda$, or of n/d; *i.e.* for a given crystal F is a function of the face from which reflexion takes place, and of the order of reflexion only, independent of the wave-length of the X-rays used *.

This means that no new points on the curve of F against $(\sin\beta)/\lambda$ can be obtained by using different values of λ so long as the same crystal is always used, but by using a different kind of crystal a different set of values of $(\sin\beta)/\lambda$ are made available, and more points on the curve can be obtained.

It is fairly obvious from the arguments of § 3 that the X-ray reflexion from crystals can give us no evidence of the relative orientations of different orbits or relative phases of electrons in different orbits. This is because for an electron in a given position with respect to the nucleus, the contribution is the same whatever atom it belongs to, on account of the phase relations between the waves scattered by each atom as a whole. But when the centres of the scattering atoms are distributed at random in space, so that the intensities and not the amplitudes of the waves scattered by the various atoms have to be added, the phase differences between the wavelets scattered by the individual electrons in a single atom are important †. The intensity scattered by an amorphous mass (or by a crystal at an angle which is not a glancing angle for reflexion from a particular face) will therefore depend on the relative orientations of the orbits and phases of the electrons in a single atom, and it is just possible that some information on these points could be obtained from experimental data of X-ray scattering by amorphous substances (possibly solutions). But the theoretical formulæ for F are rather heavy, and numerical evaluation of F for any definite atomic model would involve the evaluation of several lengthy double integrals, and until better agreement between theory and experiment has been obtained for reflexion from crystals it does not seem probable that any valuable information could be obtained from amorphous materials.

* This has already been mentioned incidentally by W. L. Bragg in a letter to 'Nature' (vol. cx. p. 148). He has also pointed out to me that the same is true of the Debye effect.

† W. L. Bragg has drawn attention to the difference between the effective value of F for an atom in a crystal and one in an amorphous mass of material (see B. J. B. i. p. 335).

CXVII. *The Spontaneous Loss of Energy of a Spinning Rod according to the Relativity Theory.* *By* A. S. Eddington, *M.A., F.R.S., Plumian Professor of Astronomy in the University of Cambridge* *.

A SPINNING rod sets up gravitational waves in the surrounding space, and the interesting question arises whether these waves will carry away the energy of rotation of the rod and gradually bring it to rest. This problem was first solved by Einstein †. An unimportant numerical slip in Einstein's result is corrected in a discussion by the author ‡, which for the most part followed similar methods. It appears that a rod of moment of inertia I with angular velocity ω will lose energy per unit time at the rate

$$\frac{32}{5} kI^2\omega^6,$$

where $k=1$ if the units are such that the constant of gravitation and the velocity of light are unity, and $k=2{\cdot}7\,.\,10^{-60}$ for C.G.S. units. In all practical cases the loss is extremely small.

On account of the picturesque interest of the problem it may be desirable to give a deduction of this result by a totally different method, which is considerably shorter than the discussions above quoted. In this method we pay no attention to the waves receding into the distance, which by an analytical fiction are supposed to bear the energy away. (The waves are real enough, but their pseudo-energy-tensor is not regarded in the relativity theory as real energy.) The loss of energy is considered from another aspect which has a certain historic interest. If gravitation is not propagated instantaneously the lag may introduce tangential components of the force opposing the motion of the system. This was the effect looked for by Laplace when he considered a finite velocity of propagation of gravitation in astronomical systems; from its non-appearance he deduced that the speed must be very much greater than that of light. Modern theory shows that the first order effect looked for by Laplace (proportional to v^2/c^2) is eliminated; but the result here found is actually the residual Laplace effect of the third order (proportional to v^6/c^6). The mutual attraction of the particles of the rod is, owing to the rotation, not exactly in the instantaneous line of the rod, and the resultant is a couple which slowly

* Communicated by the Author.

† Berlin *Sitzungsberichte*, 1916, p. 688; 1918, p. 154.

‡ Proc. Roy. Soc. 102 A. p. 268 (1922).

destroys the rotation. It will, however, be explained in due course that the present result does not solve Laplace's problem for an astronomical system, but is limited to cohesive systems.

From the well-known equation *

$$\partial \mathfrak{T}^{\nu}_{\mu}/\partial x_{\nu} = \tfrac{1}{2}\mathfrak{T}^{\alpha\beta}\partial g_{\alpha\beta}/\partial x_{\mu},$$

we have for $\mu = 4$

$$\frac{\partial \mathfrak{T}^{\nu}_{4}}{\partial x_{\nu}} = \tfrac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t}.$$

Hence, integrating over a three-dimensional region enclosing the spinning rod,

$$\frac{\partial}{\partial t}\int \mathfrak{T}^{4}_{4} dV = \tfrac{1}{2}\int \mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t} dV, \quad . \quad . \quad . \quad . \quad (1)$$

since the other terms on the left give surface integrals which vanish because the boundary does not pass through matter. Equation (1) expresses the rate of change of material energy in the region, *i. e.* in the rod which is the only material system there.

In calculating approximate values of $\partial g_{\alpha\beta}/\partial t$ we shall find it necessary to retain rather high powers of the velocities in the coefficients of periodic terms. But the same accuracy is not needed for steady factors such as $\sqrt{-g}$ or $(1-v^2)^{-\frac{1}{2}}$ which merely multiply the final result; these will be set equal to unity and $\mathrm{T}^{\alpha\beta}$ may thus be substituted for $\mathfrak{T}^{\alpha\beta}$.

We write $g_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}$ where $\delta_{\alpha\beta}$ denotes the Galilean values, and the square of the small disturbance $h_{\alpha\beta}$ is neglected. Then, as Einstein has shown, the law of gravitation will be satisfied by

$$\Box h_{\alpha\beta} = 2\mathrm{G}_{\alpha\beta} = -16\pi(\mathrm{T}_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta}\mathrm{T}). \quad . \quad . \quad . \quad (2)$$

This is the ordinary wave-equation of which the solution is given by an integral taken over the properly antedated strengths and positions of the sources, viz.

$$h_{\alpha\beta} = -4\int \left[\frac{\mathrm{T}'_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta}\mathrm{T}'}{r(1-v_r)}\right] dV'. \quad . \quad . \quad . \quad (3)$$

The square bracket indicates antedated values; r is the distance from dV' at the appropriate moment to the point where $h_{\alpha\beta}$ is to be calculated; v_r is the component velocity

* For this equation and (2) and (3) see (55·6), (57·7), and (74·71) in my 'Mathematical Theory of Relativity.' They are standard formulæ which will be found in most treatises.

of dV' towards this point. The necessity for the factor $(1-v_r)$ was pointed out by Liénard and Wiechert.

By (1) and (3) the rate of loss of energy of the rod is

$$2\iint\left\{T^{\alpha\beta}\frac{\partial}{\partial t}\left[\frac{T'_{\alpha\beta}}{r(1-v_r)}\right]-\tfrac{1}{2}T\frac{\partial}{\partial t}\left[\frac{T}{r(1-v_r)}\right]\right\}dV\,dV' \quad . \quad (4)$$

since $T^{\alpha\beta}\delta_{\alpha\beta}=T$. It will be seen that in this integral the loss is exhibited as arising from the mutual action of pairs of elements of the rod, dV and dV'.

The square brackets can be expanded in terms of contemporaneous values of r and $T'_{\alpha\beta}$ by the series*

$$\left[\frac{T'_{\alpha\beta}}{r(1-v_r)}\right]=\frac{T'_{\alpha\beta}}{r}-\frac{d}{dt}T'_{\alpha\beta}+\sum\frac{(-)^n}{n!}\frac{d^n}{dt^n}(r^{n-1}T'_{\alpha\beta}), \quad . \quad (5)$$

the quantities on the right *not* being antedated.

Let the rod, spinning in the xy-plane, be along the axis of x at the instant $t=0$, the origin being at the centre. Let the element dV then be at x and dV' at x'. The varying distance of the element dV' from the fixed point x instantaneously occupied by dV, is

$$r=(x^2+x'^2-2xx'\cos\omega t)^{1/2}.$$

This must be used in (5), and t must be made zero after the differentiations. For the present application we can simplify (5) by noting that if $T^{\alpha\beta}$ is a component which does not vanish for $t=0$, $T'_{\alpha\beta}$ will be an even function of t, so that derivatives of odd order disappear. Accordingly

$$\frac{\partial}{\partial t}\left[\frac{T'_{\alpha\beta}}{r(1-v_r)}\right]=-\frac{d^2}{dt^2}T'_{\alpha\beta}-\frac{1}{6}\frac{d^4}{dt^4}\{T'_{\alpha\beta}(x^2+x'^2-2xx'\cos\omega t)\}$$
$$-\frac{1}{120}\frac{d^6}{dt^6}\{T'_{\alpha\beta}(x^2+x'^2-2xx'\cos\omega t)^2\}-\ldots\ldots \quad (6)$$

We have then to substitute (6) in (4) and pick out the earliest non-vanishing terms. Note that terms containing an odd power of x or x' will vanish on integration along the rod, assuming it to be symmetrical.

(a) *Stress components*, T^{11}, T^{22}.

For these only the first term of (6) is required, and since r is then not involved the double integration breaks up into the product of two independent integrals. The loss of energy is

$$-2\int T^{11}dV\cdot\frac{d^2}{dt^2}\int T'_{11}dV'-2\int T^{22}dV\cdot\frac{d^2}{dt^2}\int T'_{22}dV'. \quad . \quad (7)$$

* I am not sure if this expansion is published elsewhere; a proof is given at the end of this paper.

If σ is the line-density of the rod, $T^{22}dV = \sigma\omega^2 x^2 dx$. Hence

$$\int T^{22}dV = I\omega^2.$$

The component T^{11} represents the tension in the rod and it is easily found from elementary dynamics that its integral is $-I\omega^2$.

For the moving element the corresponding integrals are $I\omega^2 \cos 2\omega t$ and $-I\omega^2 \cos 2\omega t$. Hence (7) gives the result $16\, I^2\omega^6$.

(b) *Momentum components*, T^{24}, T^{42}.

$$T^{24}dV = \sigma\omega x dx, \qquad T'_{24}dV' = -\sigma'\omega x' dx' \cos \omega t.$$

The first term of (6) now yields nothing owing to the odd powers of x and x'. We take the second term and obtain

$$-2\iint \sigma\omega x dx \,.\, \sigma'\omega x' dx' \,.\, \frac{1}{6}(2\omega)^4 x x' = -\frac{16}{3} I^2\omega^6.$$

T^{42} gives an equal contribution, making a total of $-\frac{32}{3} I^2\omega^6$.

(c) *Mass components*, T^{44}, T.

$$T^{44}dV = \sigma dx, \qquad T'_{44}dV' = \sigma' dx'.$$

The third term of (6) is now needed, giving

$$2\iint \sigma dx \,.\, \sigma' dx' \,.\, \frac{1}{120}(2\omega)^6 \,.\, 2x^2 x'^2 = \frac{32}{15} I^2\omega^6.$$

The proper-density T and the coordinate-density T^{44} are practically the same, so that the term in T takes away again half the above amount leaving $\frac{16}{15} I^2\omega^6$.

Gathering together (*a*), (*b*), and (*c*) the rate of loss of energy is

$$\left(16 - \frac{32}{3} + \frac{16}{15}\right) I^2\omega^6 = \frac{32}{5} I^2\omega^6, \quad . \quad . \quad . \quad (8)$$

agreeing with the result already stated.

If a is of the order of magnitude of the linear dimensions of the rod, and v the linear velocity, the result (8) is of order $(M/a)^2 v^6$. Since we have neglected higher powers of the gravitational potential, terms of order $(M/a)^3 v^2$ or $(M/a)^4 v^2$ have not been investigated. The result is only valid if these are small compared with (8). For a spinning rod there is no theoretical difficulty in constructing examples in which

this condition is satisfied ; but in a double star system * v^2 is necessarily of the same order of magnitude as M/a, so that the approximation is *never* permissible.

Series expansion of Retarded Potentials.

The expansion (5) is obtained as follows. Consider a fixed point P at the instant t and a moving source P′ at the instant $t-\tau$. Let the distance PP′ be given as a function of the time so that

$$PP'=r=f(t-\tau).$$

Then the component velocity of P′ along P′P is

$$v_r=-dr/dt=-f'(t-\tau).$$

Suppose that the wave emitted from P′ at the time $t-\tau$ has at the time t reached the point Q in the direction P′P. Let $PQ=\alpha$, so that

$$\alpha=\tau-r=\tau-f(t-\tau). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the wave-velocity being unity. α is then a function of τ, and *vice versa*.

By differentiation

$$1=\frac{d\tau}{d\alpha}+f'(t-\tau)\frac{d\tau}{d\alpha}=\frac{d\tau}{d\alpha}(1-v_r).$$

Hence if $\phi(t-\tau)$ is any quantity associated with the source at P′ at the time $t-\tau$,

$$\left\{\frac{\phi}{r(1-v_r)}\right\}_{t-\tau}=\frac{\phi(t-\tau)\,d\tau}{f(t-\tau)\,d\alpha}=\frac{d}{d\alpha}\mathrm{F}(t-\tau),$$

where

$$\mathrm{F}'=-\phi/f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The appropriate moment required for calculating a retarded potential is obtained by putting $PQ=\alpha=0$. Hence

$$\left[\frac{\phi}{r(1-v_r)}\right]=\left\{\frac{d}{d\alpha}\mathrm{F}(t-\tau)\right\}_{\alpha=0}. \quad . \quad . \quad (11)$$

By Lagrange's theorem on the expansion of an implicit function which by (9) satisfies

$$(t-\tau)=(t-\alpha)-f(t-\tau)$$

* The density of the rod is not assumed to be uniform, so that at first sight a double star would appear to be simply a particular case of a rod.

we have

$$\begin{aligned} F(t-\tau) &= F(t-\alpha) - F'(t-\alpha)f(t-\alpha) \\ &\qquad - \sum_{2}^{\infty} \frac{1}{n!} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} [F'(t-\alpha)\{f(t-\alpha)\}^n] \\ &= F(t-\alpha) + \phi(t-\alpha) \\ &\qquad + \sum_{2}^{\infty} \frac{1}{n!} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} [\phi(t-\alpha)\{f(t-\alpha)\}^{n-1}]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{d\alpha} F(t-\tau) &= \frac{\phi(t-\alpha)}{f(t-\alpha)} - \phi'(t-\alpha) \\ &\qquad + \sum_{2}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \alpha^n} [\phi(t-\alpha)\{f(t-\alpha)\}^{n-1}]. \end{aligned}$$

Substituting in (11) and noticing that $\partial/\partial\alpha = -\partial/\partial t$,

$$\left[\frac{\phi}{r(1-r_r)}\right] = \frac{\phi}{r} - \frac{d\phi}{dt} + \sum_{2}^{\infty} \frac{(-)^n}{n!} \frac{d^n}{dt^n}(r^{n-1}\phi), \quad . \quad (12)$$

where on the right ϕ and r are to be taken for the time t.

This result is available for calculating electromagnetic potentials irrespective of the theory of relativity as well as for the gravitational potentials in (5).

CXVIII. *The Sparking-potential of Gases.* *By* G. Holst *and* E. Oosterhuis *.

1. In his paper "Ionization by Collision" † J. S. Townsend has discussed a theory of the sparking-potential of gases suggested by us in a short note in the *Comptes Rendus* ‡. This theory is based on the assumption that positive ions can liberate electrons from the cathode, no ionization taking place by the collision of a positive ion against a gas molecule. It may be of interest to give here an account of the considerations and experimental facts underlying this hypothesis.

2. A gas molecule can be ionized by an electron as soon as the energy of the electron reaches a certain value corresponding to the ionization-potential of the molecule. In order that a positive ion may be able to ionize, it should have *at least* the same energy. In many cases the conditions in a discharge-tube are such that it is impossible for a positive ion to acquire this energy.

* Communicated by the Authors.

† J. S. Townsend, Phil. Mag. vol. xlv. p. 444 (March 1923).

‡ G. Holst and E. Oosterhuis, *C. R.* t. clxxv. p. 577 (1922).

From Gill and Pidduck's measurements* in helium we take the following data: for a pressure of 5 mm. and electrode distance ·78 cm. a sparking-potential of 156 volts was found. The mean free path being only 0·03 mm., the voltage drop per mean free path in the direction of the field is 0·6 volt. Now the ionization potential of helium amounts to about 25 volts, so that the ion should describe a path *at least* 40 times greater than the average in the direction of the field in order to ionize. The probability of describing a path of this length in an arbitrary direction being already $<10^{-17}$, it becomes evident that practically all ionization by positive ions can be neglected.

3. During a series of determinations of current-potential curves in argon and neon we had observed that in argon the probability that an electron loses its energy at a velocity below that corresponding to the ionizing potential is much larger than in the case of neon †.

Later experiments with neon have shown that in all cases where the number of collisions is not very large, practically all the electrons will ionize without losing part of their energy by excitation in a preceding collision. Ionization itself seems to be readily produced. It could therefore be expected that the phenomena in neon would show a very simple character, and for this reason most of our experiments and considerations relate to neon.

4. If an electric current passes through a tube filled with neon a glow can already be observed at very small currents of the order of magnitude of 10^{-9} amp. The appearance of this glow can give us some information about the phenomena going on during the discharge.

A neon discharge-tube with electrode distance about 1 cm., gas pressure about 1 cm., was put in series with a diode. By regulating the filament current of the latter, the current in the neon tube could be varied from very small values to a value where the cathode glow begins. So the successive forms of the discharge were studied. First a very dull glow near the anode is observed. On increasing the current a series of black and luminous layers appears, a perfectly black one just in front of the cathode followed by a very sharply limited luminous layer, etc.; the layers becoming more diffuse near the anode. At higher current this striated form

* *Cf.* J. S. Townsend, 'Electricity in Gases,' p. 325 (1915).

† G. Holst and E. Oosterhuis, *Physica*, i. p. 80 (1921); *cf.* also H. C. Rentschler, Phys. Rev. (2) xiv. p. 503 (1919).

of discharge makes place for the ordinary cathode glow ; the first black layer near the cathode persists, however *.

The fact that six or seven layers can easily be observed points out that practically all the electrons must lose their energy at the same distance from the cathode. The number of layers is equal to the potential on the tube divided by the ionizing potential. The appearance of these layers is the main basis of our hypothesis that the electrons are liberated at the cathode surface. If they were generated in the bulk of the gas it would be very difficult to explain these partly perfectly sharp layers.

Now the sparking-potential may be defined as the potential which is required to maintain a very small current †, so that we may expect that the mechanism of the sparking is analogous to that observed with small currents : the electrons liberated from the cathode ionize at definite distances from the latter.

5. In experiments with discharge-tubes containing two electrodes of different material we found ‡ that the minimum sparking-potential in neon can vary as 1:3, depending on the material of the cathode. The lowest value was found using rubidium or cæsium as cathode, the highest value was observed with a carbon cathode. If we now consider the minimum sparking-potential for different gases, the cathode being of the same metal, we find that the minimum sparking-potentials also vary about as 1:3 §. It follows that the minimum sparking-potential depends as much on the nature of the cathode material as it does on the nature of the gas.

6. The facts described above lead us to consider a hypothetical ideal gas, in which :

(1) An electron loses no energy whatsoever in collisions as long as its velocity is below that corresponding to the ionizing potential.

(2) An electron will ionize as soon as its velocity is equal to the ionization potential V_i.

(3) n positive ions will liberate one electron at the cathode surface.

(4) No positive ion ionizes a gas molecule by collision.

* G. Holst and E. Oosterhuis, *Physica*, i. p. 86 (1921) ; *cf.* F. W. Aston, Proc. Roy. Soc. (A) lxxx. p. 45 (1908).

† J. S. Townsend, 'Electricity in Gases,' p. 430 (1915).

‡ G. Holst and E. Oosterhuis, *Versl. Kon. Ak. v. Wet. Amsterdam*, xxix. p. 849 (1920).

§ J. S. Townsend, 'Electricity in Gases,' p. 329 (1915).

An electron starts at the cathode. At a distance d from the cathode—where the electron has fallen through a P.D. equal to V_i—it ionizes for the first time, forming one new electron and one positive ion. At a distance $2d$ it ionizes again, and so on. The total number of positive ions formed is $2^g - 1$, g being the number of times the process is repeated.

Sparking will occur when these ions, on reaching the cathode, liberate one electron or more, for in that case the current will continue to flow. As soon as g reaches a value g_s, given by the relation $2^{g_s} - 1 > n$, the condition for sparking is reached; the sparking-potential V_s is equal to $g_s V_i$ *.

n will be different for different cathode materials and for different ions. n will be larger if it is difficult to extract an electron from the cathode, and thus n will depend on ϕ, the work necessary to liberate an electron from the cathode. Indeed, we have shown that there is a parallelism between the sparking-potential for different cathode materials and Richardson's constant ϕ †.

For our hypothetical ideal gas the sparking-potential $V_s = g_s V_i$ would be independent of the gas pressure and the electrode distance; it depends only on the ionization potential of the gas and on the number of electrons liberated from the cathode by a positive ion.

7. Our ideal gas is only a rough first approximation. In a real gas the sparking-potential depends on gas pressure and electrode distance. This is due to the fact that in a real gas energy is lost by collisions.

We now consider a second approximation. The energy loss is limited to the loss $\Delta E = 2\frac{m}{M} E$ which the electron of mass m and energy E suffers in an elastic collision against a gas molecule of the mass M, all other energy losses being neglected as long as the velocity of the electron is smaller than that corresponding to the ionization potential.

We then find:

$$V_s = V_i \frac{273\, a\, p\, 2\sqrt{2k}}{\mathrm{T}.76.\lambda_n 4\sqrt{2}} \frac{e^{\frac{273\, a\, p\, 2\sqrt{2k}}{\mathrm{T}.76.\lambda_n g_s 4\sqrt{2}}} + 1}{e^{\frac{273\, a\, p\, 2\sqrt{2k}}{\mathrm{T}.76.\lambda_n \cdot g_s 4\sqrt{2}}} - 1},$$

* On this theory the sparking-potential is a multiple of V_i, if different cathode materials are used. Our experiments point in this direction. Compare also H. Schüler, *Phys. Zs.* xxiv. p. 259 (1923).

† G. Holst and E. Oosterhuis, *Physica*, i. p. 84 (1921).

in which

a = electrode distance (cm.);

p = gas pressure (cm. Hg);

λ_n = mean free path of a molecule of the gas at 0° C. and 76 cm.;

$k = 2\frac{m}{M}$.

The formula shows that the sparking-potential is a function of ap (Paschen's law). It holds approximately for neon near the minimum sparking-potential *. It does not hold for argon. This is to be expected, because in argon the probability of an energy loss in an exciting collision (at voltages over 12 volts) is very much larger than in neon. These energy losses may not be neglected here. In the case of argon a further approximation would be necessary to find V_s as a function of a, p, and g_s. If V_1 is the lowest potential at which an excitation can take place, V_2 the second, and so on, η_1 the probability that an electron loses the energy eV_1, η_2 the probability to lose eV_2 and so on, the average energy loss will be

$$\eta_1 n_1 V_1 e + \eta_2 n_2 V_2 e + \dots ,$$

where n_1 is the number of collisions between V_1 and V_i, n_2 between V_2 and V_i, and so on. The probabilities $\eta_1 \dots$ have not yet been determined. Moreover, the calculation of the number of collisions in argon is very difficult, as the apparent diameter of the molecule, according to Ramsauer †, is not a constant. So, even in the case of argon, no formula for V_s can be derived.

With helium a new complication is introduced by the fact that metastable states appear.

In the case of polyatomic gases new difficulties arise due to inelastic collisions and electron affinity.

The departure from the ideal gas is due to energy losses of the electrons. If the number of collisions is small these losses will be less important. That is the reason why the minimum sparking-potentials for gases are not very far apart, the difference in sparking-potential increasing with pressure and electrode distance.

* We must point out that these considerations apply only to gas pressures p and electrode distances a larger than those corresponding to the minimum sparking-potential. For smaller values of pa the number of collisions becomes too low, so that the sparking-potential increases.

† C. Ramsauer, *Phys. Zeitschr.* xxii. p. 613 (1921); G. Hertz, Proc. Amsterdam, xxv. nos. 3 & 4 (1922).

8. There is one point in our reasoning that must be considered somewhat more in detail. Our experiments showed us that the probability that excitation occurs in neon is very small. On the other hand, the *luminous* layers observed with small currents show that not only ionization occurs but also some excitation. The sharpness of the layers proves, however, that the number of collisions giving excitation is small compared with the number of ionizing collisions. This is in accordance with the fact that the amount of light per unit of current given out by the luminous layers is very much smaller than in the positive column.

Physical Laboratory,
Philips' Incandescent Lamp Works,
Eindhoven.

CXIX. *Notices respecting New Books.*

The Structure of the Atom. By E. N. da C. Andrade, D.Sc., Ph.D. (G. Bell & Co.) 16*s*.

ATOMIC Physics has been developing recently at such a great rate that those general students of science who have been unable to keep pace with the literature will heartily welcome Professor E. N. da C. Andrade's book entitled 'The Structure of the Atom.'

In a volume of only 300 pages it is hardly to be expected that a comprehensive survey of this vast subject in all its details should have been made, but the author has succeeded in so small compass in giving quite a good general survey. Its utility to the reader is enhanced by the addition at the end of each chapter of references to some of the more important original published memoirs.

The book is divided into two sections, the first dealing with the nucleus, and the second treating the structure outside the nucleus.

It naturally follows, therefore, that the eight chapters under the first heading are devoted in the main to a consideration of Radioactivity and Positive Rays, while the remaining six chapters deal with the various dynamic and static models of the atom which have been proposed in recent years and their success in correlating the known experimental data in X-rays, spectroscopy, and chemistry, and in suggesting new lines for experimental and theoretical investigations.

We sincerely congratulate the author on his very useful work and hope that another volume may soon appear.

Published by G. Bell, the book is excellently produced, easy to read, and well illustrated.

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